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THE ELEMENTS OF HIGH SCHOOL MATHEMATICS

COMPRISING ARITHMETIC, PRACTICAL GEOMETRY, AND ALGEBRA

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PREFACE

Never was the recognition of pedagogical demands so general among teachers of high school mathematics as it is today. Nor was it ever so clearly apparent as it is today that the problem of the teacher of mathematics is chiefly a problem of a two-fold adaptation of material: first, to individual needs; second, to the community need. Both needs being in continual change, the latter very markedly, reorganization is always opportune.

That we are now in a special sense in a period of reorganization is supported also by pertinent evidence. The Commission on the Reorganization of Secondary School Mathematics of the National Education Association, after three years of work, has recently published, through the United States Bureau of Education, its body of conclusions as to a plan of very radical reorganization of the courses in high school mathematics. About a year ago the National Committee on Mathematical Requirements in Secondary Schools also issued its now famous Secondary School Circular, 1920, No. 5, on the Reorganization of the Early Courses in High School Mathematics. The latter committee worked under the auspices of the Mathematical Association of America, and had in its membership a representative of every section of the nation and of every segment of the American school system. Both of these very significant reports urged thorough-going reorganization of existing courses: curtailing and eliminating much old matter and introducing much new material; but leaving the questions of form of organization and mode of presentation of the material open to individual preference. Furthermore, numerous textual forms of material for the first and second high school years have appeared in the

literature for teachers. Nearly every program of associations of mathematics teachers that has appeared in print for the last three years has contained one or more titles on some phase of proposed reorganization. The United States Bureau of Education has also declared its willingness to go as far as its funds will permit, to foster and promote reformative movements in high school mathematics. Several of our large cities are claiming to have completed a reorganization of the mathematics courses in their schools. Indeed, the burden of proof for the soundness of his position is now on the "stand-patter," who would maintain unchanged the existing order in high school mathematics.

The particular purposes of the authors of this course are:

1. To furnish a textbook that will give to all students in the first year of the four-year high school a *thorough and comprehensive, though brief,* review of the more essential portions of the arithmetic that is fundamental to high school mathematics. Fractions, common and decimal; percentage and interest, in their various applications; and certain phases of compound numbers and mensuration, are treated from a somewhat higher viewpoint than could be used in the grades. The aim in this review work is to develop *accuracy* and *speed,* as well as *facility* and *permanence* of grasp. It will be conceded generally that a mastery of these subjects will greatly benefit the young person in any branch of commercial or trade work to which he may turn; and, coming at the period when, unfortunately, so many are about to leave school for work in stores, shops, etc., this review will be particularly helpful to these pupils.

2. So to teach the necessary course in algebra as to make an easy transition from the language and methods of arithmetic, with which the student is presumed to be fairly familiar, to those of algebra. Early in the course he is made acquainted with the notation of algebra by applying *general conditions to* simple problems (see Chapter III)

and is led by easy steps to the more complicated manipulations of general terms. It is believed that the methods followed in this text, by making easier the transition from arithmetic to algebra, will save time for the overcoming of difficulties inherent in the subject. The specific aim is to make the transition from arithmetic to algebra intelligible, facile, and revealing of deepened quantitative insights.

The first, second, fifth, sixth, and seventh chapters of this book treat subjects commonly taught in grade arithmetic, but not with the same purpose nor by the same methods as are there used. In arithmetic the pupil studies to learn how to do particular problems. He tries to learn arithmetic for its own sake. Now he tries to learn it *as a basis for advance*, for the next mathematical step. Here he studies type-problems, the particular type being valuable in showing him the essential way of handling all problems of the class typified. This recognizing and stating the essential process of a problem is *generalizing*, which is algebraic study at its best. Thus the work gives specific training for skill in precisely this algebraic thing even in these seemingly arithmetical chapters. The general educational objective of the specified chapters is *to rationalize related fundamental arithmetic*, and at the same time *to give it an algebraic turn.*

The algebraic part of the book makes prominent the equation as a controlling theme, eliminates complicated factoring and fractional forms, curtails other topics that are commonly too highly elaborated, and makes much of the study of the quantitative relations in problem situations. The eliminations conform notably to the reports mentioned on page 1 of the Preface.

Model forms of solution are frequently given, and thought values are everywhere emphasized. The treatment of algebra is essentially an elementary treatment. Very great care has been given to simple, direct explanations and

statements. Formal definitions are presented in very simple language, and are used sparingly—only where there is a real gain by giving them. Graphs are employed *at the beginning* of the treatment of simultaneous equations to give insight into the significance of algebraic solutions, and not as an independent method competing with the algebraic. The treatment strives throughout to avoid excessive wordiness, though it nowhere forgets the importance of having words stand for clear ideas when they are used. Clearness, directness, and reasonable compactness of presentation have been the guiding ideals in the language of the book in general, and of the algebra in particular.

3. In the chapter on mensuration the hand and eye of the pupil are trained through the use of the simpler drawing instruments—the compass and the ruler; and in addition to the computation of various magnitudes, he is taught to derive many of the easier theorems of plane geometry, though no attempt at logical demonstration is made.

Particular attention is called to the following features:

(1) The unusually large number of exercises and problems offered for solution—more than any individual can be expected to do. This will enable the instructor to distribute the work of problem-solving among several classes or sections without the work of any two groups overlapping more than may be desirable.

(2) The careful attention given to checking throughout. This feature will enable the pupil to verify his work with such a degree of confidence that he will not feel the constant need of an answer book; although answers have been prepared and will be furnished teachers upon request.

(3) The problems that are in the main practical, providing a constant application of mathematical principles to situations of everyday life.

(4) The compact summary at the close of each chapter that affords a helpful basis for review work.

(5) The comprehensive index at the close for aid in reference and reviews.

In plan this book differs sharply from the other texts that have thus far appeared for the first high school course in response to the general purpose of reorganization. Nearly all others have attempted some sort of intermingling or blending of materials drawn from arithmetic, algebra, and geometry. The treatment offered in this book makes no especial attempt to blend or mingle the separate materials. The treatments here are in separate units, one of arithmetic, one of mensurational geometry of an experimental and inductive type, and one of elementary algebra. The book is definitely what its title claims for it, viz.: *The Elements of High School Mathematics.*

It also differs from the other recent texts in that it does not include any trigonometric material, such material not being considered essential to the *elements* of high school mathematics. This text may be regarded as presenting a type of *mixed mathematics,* so far as the entire course is concerned, but the mathematics is not mixed in treatment; the arithmetic, the geometry, and the algebra are treated each by its own specific technic and methods. This procedure will at least facilitate the teachableness of the text. The beginnings of experimental geometry and algebra, and the later phases of arithmetic are, however, developed sufficiently closely together in time to afford means of correlation, for teachers who desire them.

On the other hand, it should be noticed that such related phases of a mathematical topic are given as are organic to a strong treatment of that topic. For example, the central principles of the commercial topics of arithmetic are formulated into simple equational relations, as also are the mensurational topics. Moreover, such geometry is employed in the mensurational work as will give a rational grasp of the underlying principles and laws. In this

fashion the pupil is led to see that algebra and geometry emerge from the arithmetic and are continuous with it. The pupil also lays his algebraic foundations in tangible surroundings. Both the algebra and the geometry to come later gain by the arithmetical grounding this treatment furnishes. The three elemental disciplines of mathematics are thus linked up into a strong continuous sequence.

At the end of this course the pupil will have completed a unit of work that strengthens and advances his arithmetic, that furnishes him some vocational insights, that gives him control of elementary algebra into quadratics, and that prepares him for a good start in standard geometry. He may take up either a standard plane geometry course, or a standard second course in high school algebra. Or, if he must leave school at or before the end of the first high school year, he will have been given the largest mastery in the allotted time of the fundamental tools and technics of the three elementary branches of mathematics.

Of the elementary tools frequently advised nowadays for the beginning mathematical course, he will be lacking only in trigonometry. This element is omitted on the ground that not enough can be done with it, in the midst of so many other new things, to enable the pupil even to appreciate the power of trigonometric method. It is felt that the early work in this subject would better come in connection with a geometry course to follow this course.

Although this book is prepared primarily for one year's work, a thorough mastery of the subject-matter contained in it will enable the student to pass the examination in algebra for admission to most of our colleges and universities, while it will in all cases give credit for one entrance unit, without examination, for admission to those institutions admitting students on high school certificates.

June, 1921 THE AUTHORS

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THE ELEMENTS OF HIGH SCHOOL MATHEMATICS

CHAPTER I

COMMON FRACTIONS

DEFINITIONS

1. Unit. A single thing is called a unit.

2. Integer. An integer is a unit or a collection of units, like 5, 7, 15, etc.

3. Kinds of Numbers. Numbers are of two kinds, viz., (*a*) Concrete* numbers; (*b*) Abstract numbers.

(*a*) Concrete numbers designate some particular set or kind of objects: as 5 sheep, 10 apples, 6 bushels of corn, etc.

(*b*) Abstract numbers are numbers which may be applied to any class of objects at will: as 5, 7, 15, etc.

4. The Fraction Idea. When an object is broken into a number of equal pieces, any one piece or combination of pieces is known as a *fraction* of the object so broken up.

For example, if an apple is cut into 7 equal pieces, then one of these pieces is called one-seventh of the apple, two pieces are two-sevenths, etc.

If each one of several apples is cut into 7 equal pieces, there may be taken up of these exactly 7 pieces, or even more than 7 pieces. This means that the group of pieces is equivalent to exactly *one* apple if 7 are taken up, or to *more* than *one* apple if more than 7 pieces, say 9 pieces, are taken up. These groups of parts are still called fractions, and are named seven-sevenths apples, or nine-sevenths apples.

* Strictly speaking, all numbers are abstract; but it is convenient to call numbers whose units are named, *concrete numbers*.

5. Proper and Improper Fractions. When the pieces of the divided object that are taken up amount to less than one unit of the kind considered, the fraction is said to be a *proper* fraction.

If the number of pieces taken up is enough to make one unit, or more than one unit of the kind considered, then the fraction is said to be an *improper* fraction.

For example, seven-sevenths, nine-sevenths, etc., are *improper* fractions.

6. Parts of a Fraction. The number that tells into how many pieces the unit is divided is called the **denominator** (*namer*) of the fraction, while the number which tells how many of these pieces are to be taken is called the **numerator** (*numberer*) of the fraction. The numerator and the denominator of a fraction are called the *terms* of the fraction.

7. Fractions are written with the numerator above the denominator and with a horizontal line between them.

For example, five-sevenths is written $\frac{5}{7}$, three-eighths is written $\frac{3}{8}$, eleven-sevenths is written $\frac{11}{7}$, etc.

8. Mixed Numbers. An improper fraction may be written as a combination of a whole number and a fraction.

For example, the improper fraction $\frac{11}{7}$ may be written $1\frac{4}{7}$, meaning one unit and a proper fraction $\frac{4}{7}$. An improper fraction, when written thus is called a *mixed number*.

Read the following: $\frac{1}{5}$, $\frac{3}{7}$, $\frac{79}{101}$, $3\frac{2}{5}$, $\frac{18}{81}$, $\frac{81}{18}$, $4\frac{7}{4}$, $14\frac{1}{4}$, $\frac{23}{5}$, $\frac{7}{4}$. Which of the above are proper fractions? Which are mixed numbers? Which are improper fractions?

REDUCTION OF FRACTIONS

9. Changing the form of a fraction *without changing its value* is called *reduction of fractions.*

A fraction is changed to higher terms if, without changing

the value of the fraction, its numerator and denominator are expressed by larger numbers. The following principle applies in all operations with fractions:

10. Principle. *The value of a fraction is unchanged if the numerator and denominator are both multiplied by the same number.*

Exercise 1

Reduce:

1. $\frac{1}{2}$ to fourths.	6. $\frac{25}{2}$ to 10ths.
2. $\frac{7}{25}$ to hundredths.	7. $\frac{1}{2}$ to 10ths.
3. $\frac{1}{4}$ to hundredths.	8. $\frac{20}{42}$ to 84ths.
4. $\frac{13}{19}$ to 38ths.	9. $\frac{25}{49}$ to 343ds.
5. $\frac{42}{20}$ to 40ths.	10. $\frac{2}{3}$ to 12ths.

11. Lower Terms. A fraction is changed to *lower terms* if, without changing the value of the fraction, its numerator and denominator are expressed by smaller numbers than the terms of the given fraction.

12. Checking. Checking is showing that a result is probably correct. A problem is not usually considered solved until it is checked.

Exercise 2

1. Check the examples of Exercise 1 by reducing to lower terms.

2. Change as directed, then check: $\frac{660}{880}$ to 88ths, to 44ths, to 8ths, to 11ths.

3. Change and check: $\frac{16}{64}$ to 32ds, 16ths, 8ths, 4ths.

13. Common Divisors. A *common divisor* of two or more numbers is a number that will exactly divide each of them.

For example, 2 is a common divisor of 6 and 28; 7 is a common divisor of 21, 91, and 343.

Exercise 3

1. What is the greatest common divisor (g. c. d.) of two or more numbers?
2. Is 2 the greatest number that will divide 6 and 28?
3. When is a fraction said to be reduced to *lowest* terms?
4. Can a fraction be reduced to *highest* terms?

Exercise 4

Find the g. c. d. of the following:

1. 56 and 49.
2. 1024 and 56.
3. 729 and 108.
4. 625, 250, and 95.

REDUCTION OF MIXED NUMBERS

The mixed number $13\frac{2}{3}$ means $13+\frac{2}{3}$. But 13 units equal $\frac{39}{3}$, therefore $13\frac{2}{3}=13+\frac{2}{3}=\frac{39}{3}+\frac{2}{3}=\frac{41}{3}$. Similarly, $7\frac{2}{5}=7+\frac{2}{5}=\frac{35}{5}+\frac{2}{5}=\frac{37}{5}$, and $15\frac{3}{4}=15+\frac{3}{4}=\frac{60}{4}+\frac{3}{4}=\frac{63}{4}$. Hence the following rule:

14. Rule. *A mixed number is reduced to an improper fraction by multiplying the integral part by the denominator of the fraction, adding the numerator to this product, and writing the sum over the given denominator.*

If 41 be divided by 3 the quotient is 13 and the remainder 2. From § 14, $\frac{41}{3}=13\frac{2}{3}$, $\frac{37}{5}=7\frac{2}{5}$, $\frac{63}{4}=15\frac{3}{4}$.

An examination of this leads to another rule.

15. Rule. *An improper fraction is reduced to a mixed number or an integer by dividing the numerator by the denominator, writing the quotient as the integral part, and writing the remainder, if any, over the denominator for the fractional part.*

Exercise 5

Reduce to improper fractions:

1. $2\frac{1}{2}$
2. $3\frac{6}{7}$
3. $19\frac{2}{5}$
4. $17\frac{13}{19}$
5. $247\frac{81}{93}$
6. $84\frac{21}{22}$
7. $171\frac{10}{41}$
8. $9\frac{7}{9}$

Reduce to mixed numbers:

9. $\frac{91}{7}$
10. $\frac{18}{5}$
11. $\frac{231}{12}$
12. $\frac{67}{16}$
13. $\frac{175}{30}$
14. $\frac{1024}{80}$
15. $\frac{1728}{231}$
16. $\frac{39}{12}$

ADDITION AND SUBTRACTION OF FRACTIONS

16. Similar Fractions. Like integers, fractions may be combined by addition, subtraction, multiplication, and division. The pupil will recall that if numbers are to be added they must be numbers of the same kind, *i.e.*, of the same name. For example, the sum of five apples and three plums cannot be expressed in a single name, until both apples and plums are given the same name, say *things*. Then we can say 5 things and 3 things are 8 things, or even 5 apples and 3 plums are 8 things. Similarly, the sum of $\frac{1}{2}$ and $\frac{1}{4}$ cannot be expressed in a single fraction until both are expressed in the same unit, or until both are of the same name, or in other words, until both have the same denominator.

Fractions that have the same denominator are called *similar fractions*.

17. Rule. *To add fractions reduce them to similar fractions with least common denominator, add the numerators, and write the sum over the common denominator.*

Rule. *To subtract one fraction from another change them to similar fractions with least common denominator, subtract one numerator from the other, and write the difference over the common denominator.*

To add or to subtract mixed numbers reduce them to improper fractions and proceed as above.

Exercise 6

Combine as indicated and check by going over the work carefully:

1. $\frac{1}{3}+\frac{1}{2}$
2. $\frac{2}{3}-\frac{1}{2}$
3. $\frac{1}{4}+\frac{1}{3}$
4. $\frac{1}{3}-\frac{1}{4}$
5. $\frac{1}{3}+\frac{1}{4}+\frac{1}{2}$
6. $\frac{4}{3}+\frac{4}{5}$
7. $\frac{3}{2}-\frac{1}{4}$
8. $\frac{11}{21}+\frac{3}{7}$
9. $3\frac{1}{7}+1\frac{1}{3}$
10. $3\frac{4}{7}+1\frac{6}{7}$
11. $271\frac{19}{25}-109\frac{10}{16}$
12. $69\frac{5}{6}+1\frac{7}{8}-50\frac{8}{9}$
13. $\frac{131}{46}+\frac{21}{23}+6\frac{5}{10}$
14. $\frac{13}{11}+\frac{55}{44}+\frac{11}{13}-\frac{44}{55}$
15. $19\frac{3}{5}+\frac{27}{8}-\frac{17}{20}$

Arrange following groups of fractions according to size:

16. $\frac{11}{12}, \frac{21}{24}, \frac{17}{18}$
17. $\frac{17}{14}, \frac{31}{28}, \frac{10}{7}$
18. $\frac{5}{3}, \frac{5}{35}, \frac{7}{50}$
19. $1\frac{6}{7}, 2\frac{2}{7}, 1\frac{16}{7}, 2\frac{9}{7}$

MULTIPLICATION OF FRACTIONS

18. Observe the following carefully:

One half of 6 horses is 3 horses, or $\frac{1}{2}\times 6=3$.
One third of 12 dollars is 4 dollars, or $\frac{1}{3}\times 12=4$.
Two thirds of 12 dollars is 8 dollars, or $\frac{2}{3}\times 12=8$.
Two fifths of 15 acres is 6 acres, or $\frac{2}{5}\times 15=6$.
One half of 6 sevenths is 3 sevenths, or $\frac{1}{2}\times\frac{6}{7}=\frac{3}{7}$.
One third of 12 fifths is 4 fifths, or $\frac{1}{3}\times\frac{12}{5}=\frac{4}{5}$.
Two thirds of 12 fourths is 8 fourths, or $\frac{2}{3}\times\frac{12}{4}=\frac{8}{4}$.
Two fifths of 15 eighths is 6 eighths, or $\frac{2}{5}\times\frac{15}{8}=\frac{6}{8}$.

The word *of* as used above means *multiplication.*

Since $\frac{3}{7}=\frac{6}{14}$, $\frac{4}{5}=\frac{12}{15}$, $\frac{8}{4}=\frac{24}{12}$, $\frac{6}{8}=\frac{30}{40}$, we can write:
$\frac{1}{2}$ of $\frac{6}{7}=\frac{6}{14}$, $\frac{1}{3}$ of $\frac{12}{5}=\frac{12}{15}$, $\frac{2}{3}$ of $\frac{12}{4}=\frac{24}{12}$, $\frac{2}{5}$ of $\frac{15}{8}=\frac{30}{40}$.
Hence the following rule:

19. Rule. *To multiply two or more fractions, multiply their numerators and divide the result by the product of their denominators.*

20. An integer can be multiplied by a fraction by the above rule, for every integer can be regarded as a fraction whose denominator is 1.

Thus, $\frac{11}{17}$ of $2=\frac{11}{17}$ of $\frac{2}{1}=\frac{22}{17}$.

21. Cancellation. Cancellation of like factors in the numerators with like factors in the denominators should be used whenever possible.

For example: $$\frac{2}{\underset{\checkmark}{5}}\times\frac{\overset{\checkmark}{4}}{\underset{\checkmark}{7}}\times\frac{\overset{\checkmark}{5}}{\underset{3}{12}}\times\frac{\overset{2}{14}}{17}=\frac{4}{51}$$

Notice the $\checkmark$ which is used in place of drawing a line through the factors canceled. There is more probability of an error being made with the use of the line because the figures are distorted by its use. This is more evident in writing than in printing. Compare the above example with the form below, both being done in your own handwriting.

$$\frac{2}{\cancel{5}}\times\frac{\cancel{4}}{\cancel{7}}\times\frac{\cancel{5}}{\underset{3}{\cancel{12}}}\times\frac{\overset{2}{\cancel{14}}}{17}=\frac{4}{51}$$

The pupil should see that canceling a factor from numerator and denominator really means *dividing both terms* by that factor. Hence he should see that 1 (not 0) is supposed to stand wherever a factor entirely disappears. Then he will not go wrong in the final multiplication of the factors remaining in numerator and in denominator. Wherever the little checks, $\checkmark$, stand, 1 should be thought as standing.

Exercise 7

1. $\frac{1}{6}\times\frac{2}{3}\times\frac{5}{7}\times\frac{4}{9}=?$
2. $\frac{121}{240}\times\frac{260}{910}\times\frac{12}{33}=?$
3. $\frac{640}{987}\times\frac{141}{370}\times\frac{84}{87}=?$
4. $\frac{34}{36}\times\frac{12}{17}\times 6=?$
5. $\frac{4}{5}\times 5\frac{1}{2}\times\frac{2}{11}=?$
6. $3\frac{1}{3}\times 2\frac{2}{3}\times\frac{6}{7}=?$
7. $\frac{42}{50}\times\frac{11}{14}=?$
8. $\frac{1024}{1728}\times\frac{144}{64}\times\frac{17}{26}=?$
9. $\frac{2}{7}\times\frac{3}{5}\times\frac{123}{119}=?$

If you ever had the idea that multiplication increases the number multiplied get rid of the idea. $\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$ is as much a multiplication problem as $4\times 7=28$. The product $\frac{3}{8}$ is less than either of the factors $\frac{3}{4}$ or $\frac{1}{2}$. Whenever a multiplier is a proper fraction, multiplication diminishes. If the multiplier is 1 there is no change in the multiplicand at all. The point to remember is that multiplication may increase or decrease, or not change the multiplicand.

22. Checking. If you have used cancellation wherever possible in the above examples check your work in each case by multiplying without cancellation and reducing the fraction thus obtained to its lowest terms. The result ought to be the same in each case. Are there any examples in which cancellation cannot be used? If so, check your work by doing it again independently.

Checking means more than getting results right. It trains the pupil to become self-critical and self-reliant. No expert ever thinks of omitting checking. The more expert he is the more checks he knows and uses. In arithmetic the pupil should learn how to know for himself that his work is right. Real knowledge is the kind that knows what it knows.

23. In multiplying a **mixed number by a whole number** it is generally easier to multiply the integer and fraction separately and add the results.

For example: $3\frac{4}{7}\times 9=27+\frac{36}{7}=27+5\frac{1}{7}=32\frac{1}{7}$,
instead of $3\frac{4}{7}\times 9=\frac{25}{7}\times 9=\frac{225}{7}=32\frac{1}{7}$.

Exercise 8

Multiply the following and check by using a different method:

1. $11\frac{3}{8}\times 14$	4. $225\frac{15}{44}\times 49$	7. $16\frac{24}{56}\times 30$
2. $14\frac{5}{9}\times 109$	5. $28\frac{6}{7}\times 37$	8. $339\frac{9}{11}\times 176$
3. $41\frac{3}{10}\times 41$	6. $39\frac{20}{81}\times 36$	9. $\frac{84}{6}\times\frac{12}{27}\times\frac{81}{98}$

DIVISION OF FRACTIONS

24. Division is the inverse of multiplication. For example, to divide 12 by 3 we find what the number 3 must be multiplied by to give 12. Since 3×4 equals 12, then 12 divided by 3 equals 4. Division is indicated either by the sign $\div$ or by writing one number over the other.

Thus: $8\div 2$ means the same as $\frac{8}{2}$.

If $3\times 4=12$, then $12\div 3=4$, and $12\div 4=3$.
If $5\times 2=10$, then $10\div 2=5$, and $10\div 5=2$.
If $\frac{1}{2}$ of $6=3$, then $3\div\frac{1}{2}=6$, and $3\div 6=\frac{1}{2}$.
If $\frac{1}{3}\times 12=4$, then $4\div 12=\frac{1}{3}$, and $4\div\frac{1}{3}=12$.
If $\frac{2}{5}\times 15=6$, then $6\div 15=\frac{2}{5}$, and $6\div\frac{2}{5}=15$.
If $\frac{2}{3}\times 12=8$, then $8\div 12=\frac{2}{3}$, and $8\div\frac{2}{3}=12$.
If $\frac{1}{2}\times\frac{6}{7}=\frac{3}{7}$, then $\frac{3}{7}\div\frac{6}{7}=\frac{1}{2}$, and $\frac{3}{7}\div\frac{1}{2}=\frac{6}{7}$.
If $\frac{1}{3}\times\frac{12}{15}=\frac{4}{15}$, then $\frac{4}{15}\div\frac{1}{3}=\frac{12}{15}$, and $\frac{4}{15}\div\frac{12}{15}=\frac{1}{3}$.

It is evident from these examples that in each case of division by a fraction the result could be obtained by inverting the terms of the fraction and multiplying.

Thus:

$$3\div\frac{1}{2}=3\times\frac{2}{1}=\frac{6}{1}=6.$$
$$4\div\frac{1}{3}=4\times\frac{3}{1}=\frac{12}{1}=12.$$
$$8\div\frac{2}{3}=8\times\frac{3}{2}=\frac{24}{2}=12.$$
$$6\div\frac{2}{5}=6\times\frac{5}{2}=\frac{30}{2}=15.$$
$$\frac{3}{7}\div\frac{6}{7}=\frac{3}{7}\times\frac{7}{6}=\frac{3}{6}=\frac{1}{2}.$$
$$\frac{4}{5}\div\frac{1}{3}=\frac{4}{5}\times\frac{3}{1}=\frac{12}{5}.$$

Hence the following rule:

25. Rule. *To divide any number by a fraction invert the terms of the divisor fraction and then multiply.*

Exercise 9 — Examples and Problems

Let the pupil notice that division may either mean increase or decrease, and that if the divisor is 1, there is no change at all.

1. $\frac{3}{4} \div \frac{1}{2} = ?$
2. $\frac{3}{4} \div \frac{3}{2} = ?$
3. $\frac{2}{3} \div \frac{6}{7} = ?$
4. $\frac{14}{27} \div \frac{7}{9} = ?$
5. $\frac{17}{42} \div \frac{9}{5} = ?$
6. $\frac{39}{19} \div \frac{13}{16} = ?$
7. $2\frac{1}{3} \div \frac{2}{3} = ?$

Reduce all mixed numbers to improper fractions, or, if it is easier, divide the whole number and fraction separately and add the results.

8. $6\frac{1}{4} \div \frac{31}{3} = ?$
9. $9\frac{2}{4} \div \frac{17}{5} = ?$
10. $3\frac{3}{5} \div 17\frac{3}{10} = ?$
11. $\frac{1}{4} \div 9 = ?$
12. $\frac{9}{5} \div 9 = ?$
13. $\frac{9}{5} \div 5 = ?$
14. $\frac{8}{17} \div \frac{1}{51} = ?$
15. $\frac{3}{5} \times \frac{4}{15} \div \frac{7}{9} = ?$
16. $\frac{31}{42} \times \frac{9}{16} \div \frac{13}{24} = ?$
17. $\frac{6}{11} \times \frac{16}{34} \div \frac{5}{6} = ?$
18. $28\frac{3}{5} \times 1\frac{2}{7} \div \frac{37}{25} = ?$
19. $16\frac{9}{10} \times 2\frac{1}{23} \div \frac{1}{3} \times \frac{1}{4} = ?$
20. $1\frac{7}{10} \times 21\frac{12}{17} \div 2\frac{1}{9} \times 13\frac{1}{3} = ?$

21. If $7\frac{1}{7}$ lb. of sugar can be bought for 50 cents, what is the price per pound?

Model Solution: The number of pounds times the cost per pound equals the total cost. Therefore

$7\frac{1}{7}$ times the cost per pound $= 50$

Hence the cost per pound $= 50 \div 7\frac{1}{7} = 50 \div \frac{50}{7} = 50 \times \frac{7}{50} = 7$. The cost per pound is 7 cents.

22. The expression "the cost per pound" is rather long to write many times. Study the following solution carefully.

Second Model Solution: Let c stand for the expression "the cost per pound." Then

$$7\tfrac{1}{7}c=50, \text{ and } c=50\div7\tfrac{1}{7}=50\div\tfrac{50}{7}=50\times\tfrac{7}{50}=7.$$

Hence the cost per pound is 7 cents.

The second solution, even in a very simple problem, is more compact than the first. It will be necessary, in order to master the more difficult problems which are to follow, that the student learn to use the shorter solution. It is good practice at first to solve the problem by the first model and then pick out some expression like the cost per pound, rate per hour, etc., which occurs often in the solution and abbreviate it by a single letter, such as c or r. The second solution will differ from the first only in the abbreviation of terms.

23. Perkins and Company sold $364\tfrac{1}{2}$ lb. of meat to a Cleveland firm for $\$109\tfrac{1}{2}$. What was the price per pound to the nearest cent?

24. The L. C. Murdock Milling Company sold a car of flour weighing 27,500 lb. for $1650. What was the price per hundred?

25. The water flows from one faucet at the rate of $28\tfrac{1}{2}$ gal. in 9 minutes, from another at the rate of 33 gal. in 10 minutes. Which flows the faster?

26. A tank holds 1500 gal. It can be filled through a pipe in $13\tfrac{2}{3}$ hours. When the tank is being filled how many gallons go through the pipe in one hour?

27. If it could be emptied in $7\tfrac{2}{3}$ hours, how many gallons go through the emptying pipe in one hour?

28. If the tank is full and both pipes are open, one filling and one emptying, how long before it will be empty?

29. In four farms there are 320 acres. The first and second are the same size. The third is twice the second and the fourth is twice the third. How many acres in each?

Let x stand for the number of acres in the first.
Then $x+x+2x+4x=320$, or $8x=320$.

30. Divide 78 cattle between two farms so that one has 5 times as many cattle as the other.

31. After selling $\frac{1}{2}$ of his land and then $\frac{3}{4}$ of the remainder Mr. Wilson still has 400 acres left. How much did he sell?

32. A man owes $7050. By saving $\frac{1}{3}$ of his salary for 9 years he can pay his debt and have $150. What is his salary?

33. A bushel of wheat makes 40 lb. of flour, and a miller takes $\frac{1}{6}$ for toll. How many bushels are required to give the grower 4 sacks (49 lb. each) of flour?

34. The sum of two numbers is 15 and one of them is the difference between $17\frac{4}{10}$ and $16\frac{2}{5}$. What is the other number?

Exercise 10 — General Problems

We give below a comparison of common weights and measures for use in this and subsequent chapters:

1 bushel (dry) (32 qt.) contains	2150.4 cu. in.
1 gallon (liquid) contains	231 cu. in.
1 cu. ft. water weighs (approx.)	$62\frac{1}{2}$ lb.
1 gallon water weighs (approx.)	$8\frac{1}{3}$ lb.
1 cu. ft. water is (approx.)	$7\frac{1}{2}$ gal.
1 English pound is	$4.8665

1. A man receives $2.75 per day. He spends $.75 per day for carfare, board, and incidentals. At the end of 31 days he has $58. How many days has he been idle, assuming that he spends nothing while idle?

Let d stand for the number of days he worked. Then $31-d$ must represent the number of days he was idle. He saved each day $2.00. Therefore,

$$2d=58$$
$$d=29$$
$$31-d=2 \text{ (days he was idle).}$$

2. A bank sells an American Banker's check on London of face value £60 for \$300. How much profit was made?

3. How much fertilizer will be needed for $5\frac{1}{4}$A, allowing 375 pounds to the acre?

4. A farm wagon bed is 9 ft. 2 in. long, $3\frac{1}{2}$ ft. wide, and 14 in. deep. How many bushels of wheat will it hold?

5. A cubic foot of ice weighs $57\frac{3}{8}$ pounds. An ice wagon carries 10 blocks 3 ft. by $1\frac{1}{2}$ ft. by 1 ft. The load brings \$9.58. Find the price per hundred pounds.

6. In a rainfall of 3 in. how many gallons of water will fall on top of a building 100 ft. by 50 ft.? How much would the water weigh?

7. How long will it take to count a million at the rate of 120 per minute?

COMPLEX FRACTIONS

26. If the several fractions or mixed numbers combined by + or − signs are divided by a similar expression, the quotient is a complex fraction.

The best way to simplify such a fraction is to combine the terms of the numerator and denominator separately into a single fraction, then invert the lower fraction according to the rule for division of fractions. Thus:

$$\frac{2\frac{1}{3}+7\frac{3}{8}-\frac{4}{5}}{\frac{4}{7}-2\frac{5}{6}+16\frac{17}{21}}=\frac{\frac{7}{3}+\frac{59}{8}-\frac{4}{5}}{\frac{4}{7}-\frac{17}{6}+\frac{353}{21}}=\frac{\frac{280}{120}+\frac{885}{120}-\frac{96}{120}}{\frac{24}{42}-\frac{119}{42}+\frac{706}{42}}=$$

$$\frac{\frac{1069}{120}}{\frac{611}{42}}=\frac{1069}{120}\times\frac{42}{611}=\frac{1069}{20}\times\frac{7}{611}=\frac{7483}{12220}.$$

Exercise 11

Simplify:

1. $\dfrac{144\frac{1}{2}+1\frac{1}{3}}{376\frac{8}{9}-1\frac{1}{8}}$

2. $\dfrac{276\frac{2}{5}-1\frac{1}{6}}{146\frac{3}{4}-24\frac{11}{24}}$

3. $\dfrac{36-33\frac{1}{3}}{39\frac{3}{4}-17\frac{2}{3}-12\frac{5}{8}}$

4. $\dfrac{19\frac{2}{3}+\frac{5}{6}+42\frac{8}{11}}{27\frac{3}{7}-\frac{9}{13}}$

5. A man has $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{5}{7}$ of his money left. What fraction of his money is left?

6. If the man of problem 5 had $12 left, how much had he at first?

7. A merchant sells $\frac{2}{3}$ of his shoes at ordinary sales, $\frac{1}{2}$ of the remainder at special sales, $\frac{2}{5}$ of the remainder at bargain counter. What fraction of the original stock is left?

8. If there are 72 pairs left how many had he at first?

27. Summary of Chapter I. In this chapter we have reviewed how to add, subtract, multiply, and divide fractions. The most important of the rules are collected here for convenience.

The value of a fraction is not changed by multiplying both terms by the same number. § 10.

To add several fractions or to subtract one fraction from another reduce them to a common denominator and perform the indicated operations on the numerators, writing the result over the common denominator. § 17.

To multiply two or more fractions together multiply the numerators for a new numerator and multiply the denominators for a new denominator. Cancel whenever possible. § 19.

To divide one fraction by another invert the terms of the divisor fraction and proceed as in multiplication. § 25.

Frequently several of these operations are combined in one problem.

In addition to the four fundamental operations shortened notation was introduced in Exercise 9 which will greatly aid our advance in later chapters.

28. The first ten exercises below are for practice in the manipulation of fractions. The student is warned against two very common errors:

1. Never cancel a *term* of the numerator with a *factor* of

the denominator. *Only factors of both numerator and denominator can be canceled.*

2. If plus and minus signs are used together with multiplication and division signs, the latter operations are performed in the order that they occur and always take precedence over the former. That is,

$$\begin{array}{llllll} 6-18\div 2+24\div 6\times 3+7\times 3= & & & & & \\ 6-9 & +\ 4 & \times 3+21 & = & & \\ 6-9 & + & 12 \quad +21 & = & 30 & \end{array}$$

Exercise 12 — Review

Simplify the following:

1. $\dfrac{\left(\frac{6}{4}+18+\frac{28}{2}\right)}{\frac{512}{14}-17}$

2. $48\times 6\div 12-10\div 2\times 6-1$

3. $56-24+16\div 2-16\times 3+8-85\div 17$

4. $\frac{2}{3}+\frac{5}{6}-\frac{3}{8}+\frac{18}{81}\times\frac{9}{15}\div\frac{5}{6}$

5. $\dfrac{4\times 18\times 9-48+39}{12\times 18\times 5}$

6. $4\times 18\div 12\times 9-48+39\div 12\times 5\div 18$

7. $\dfrac{218748-114291}{1728-79+148}$

8. $\frac{6748}{6674}\div\frac{18225}{14070}\times\frac{36741}{1985}$

9. $2\frac{3}{4}+\frac{5}{2}$ of $\dfrac{8}{2\frac{1}{3}}+\dfrac{\frac{3}{7}}{\frac{9}{14}}$

10. $\frac{15}{16}-\frac{7}{8}\times\frac{19}{21}\div\frac{3}{7}+\frac{13}{15}$ of $\frac{225}{169}\div\frac{15}{25}$

11. A, B, C, and D own a bank capitalized at \$25,000. A owns $\frac{1}{5}$ of the stock, B owns $\frac{2}{5}$, C owns $\frac{6}{25}$. What part does D own? How much is D's part worth at par? How many shares does D own if each share is \$25?

12. In problem 11, what part of the bank do A and B together own? A and C? Can any two of them control a stockholder's election?

13. Herbert, James, and John together have 42 marbles. James has twice as many as Herbert, while John has 3 times as many. How many has each?

Let n denote number Herbert has.

14. In problem 13, is it possible for James and John each to have twice as many as Herbert? Why?

15. In problem 13, is it possible for James and John each to have 3 times as many as Herbert? Why?

16. A man purchased a number of sheep for \$168, paying an average of \$$4\frac{1}{5}$ per head. How many did he purchase?

17. The sheep in problem 16 weighed 8600 lbs. What was the average weight and what was the cost per pound?

18. Divide 1728 into two numbers such that 9 times the smaller equals 5 times the larger.

19. A cubic foot of steel and a cubic foot of water together weigh $552\frac{1}{2}$ lb. The difference in their weight is $427\frac{1}{2}$ lb. Find the weight of each. How many times the weight of the water equals the weight of the steel?

20. In an examination of 15 problems each pupil received 8 points for each problem solved correctly; 4 points were taken off for each one of all the other problems. A student scored 60 points. How many did he solve?

21. One ounce of chocolate is equivalent in flavoring strength to $\frac{1}{8}$ a box of cocoa. If chocolate costs 45 cents a pound (16 oz.) and cocoa 25 cents per box, which is the cheaper?

22. Potatoes pared and then boiled lose $\frac{1}{40}$ of their starch. Potatoes boiled with the skins on lose $\frac{1}{50}$ of their starch. Find what fraction of the starch may be saved by boiling potatoes with the skins on.

23. Sea water is 1.024 times as heavy as fresh water. How much difference is there in the weight of 10.000 gallons of each?

CHAPTER II

DECIMAL FRACTIONS. METRIC SYSTEM

DEFINITIONS

29. Power of a Number. If a number is multiplied by itself and the product is multiplied by the same number again, etc., each of the products thus obtained is said to be a *power* of the number.

For example, 5×5 ($=25$) is called the *second power* (or square) of 5; 5×25 ($=125$), the *third power* (or cube) of 5; 5×125 ($=625$), the *fourth power* of 5, etc.

30. Decimal Fraction. A *decimal fraction* is a fraction in which the denominator is some power of ten.

For example, $\frac{3}{10}$, $\frac{37}{100}$, $\frac{36}{1000}$, $\frac{153}{10000}$, etc., are decimal fractions.

31. Writing Decimal Fractions. Decimal fractions are more conveniently managed than common fractions, because of the simplicity of the denominators. The denominators of decimal fractions are usually not written, but indicated by the *decimal point.*

For example, the fraction $\frac{3}{10}$ is written .3. The period, called the decimal point of the fraction, takes the place of the denominator. Similarly, $\frac{3}{100}$ is written .03, $\frac{15}{1000}$ is written .015, $\frac{3}{1000}$ is written .003, etc.

When the fraction is written in this way the part to the right of the decimal point must contain as many figures as there would be zeros in the denominator, if it were written. If the part to the right of the decimal point does not contain so many figures then enough zeros are placed immediately

following the decimal point to make the required number of places, or figures.

For example, $\frac{3}{1000}$ is written .003, the two zeros being placed after the decimal point so that there will be three figures in the fraction, corresponding to the three zeros in the denominator.

In the case of a mixed number (whole number and fraction), the whole number is written to the left of the decimal fraction.

For example, $3\frac{3}{100}$ is written 3.03

ADDITION OF DECIMALS

32. The operations upon decimal fractions, *i.e.*, multiplication, addition, subtraction, and division, are done in the same way as with common fractions, except that, since the denominators are always powers of 10, certain simplifications occur.

33. A decimal fraction can be changed to *higher terms* by simply annexing zeros after the last significant figure, since that is equivalent to multiplying both numerator and denominator by 10.

For example, $.13 = \frac{13}{100} = \frac{130}{1000} = .130$

To add decimals reduce them to the same order (*i.e.*, to a common denominator) and add as common fractions, thus

$$31.73 + 12.346 = 31.730 + 12.346 =$$
$$\frac{31730}{1000} + \frac{12346}{1000} = \frac{44076}{1000} = 44.076$$

The same result can be obtained by the following rule:

34. Rule. *Write one number under the other, being careful to place the decimal point of one directly under that of the other. Then add as with integers.* Thus,

$$\begin{array}{r} 31.73 \\ \underline{12.346} \\ 44.076 \end{array}$$

For addition it is not even necessary to write the zero after the 3 since the result is not changed by it.

In the following exercises add up the columns first and then check by adding down. Bankers, bookkeepers, and business men check their results in this manner.

Exercise 13

1.	2.	3.
7317.4512	2715.84	6.41
8476.13	9017.4389	9416.77
80.767	1622.99	2000.059
931.21634	11.832	104.0078
3.84	.04751	161.5516
	9417.236	7041.37

4. Add and check the columns in the day book below:

Ledger		*Cash*		*Bank*	
Dr.	Cr.	Dr.	Cr.	Dr.	Cr.
135.78					135.78
	4.50	4.50			
	505.00	505.00			
			509.50	509.50	
8.00	5.00	5.00			8.00
2.16	103.75	103.75			
			108.75	108.75	
150.00					150.00
255.89					255.89

5. Check the debits against the credits above.

SUBTRACTION OF DECIMALS

35. To find the difference between two decimal fractions write them with the subtrahend placed beneath the minuend,

with decimal points arranged as for addition, then subtract as with whole numbers.

For example: From 707.5162 subtract 221.03256.

Operation: Write the numbers as follows:

$$\begin{array}{r} 707.5162 \\ \underline{221.03256} \\ 486.48364 \end{array}$$

36. Rule. *Write the numbers of the subtrahend beneath the minuend with the decimal point of one directly under that of the other, begin at the right, and subtract as in integers.*

Exercise 14

Subtract and check by adding the subtrahend to the difference:

1. 27.045 16.943	**3.** 92.67544 89.03749	**5.** .00671 .000671
2. 1.0003 .9996	**4.** 60. .00395	**6.** 16741.95 11947.365

REDUCTION OF DECIMALS TO COMMON FRACTIONS

37. It is often convenient to change a decimal into a common fraction. Since a decimal fraction differs from a common fraction only in having 10 or a power of 10 for its denominator, this change can be made by the following rule:

38. Rule. *To reduce a decimal to a common fraction omit the decimal point, write the denominator, and reduce the fraction to its lowest terms.*

For example: $.75 = \frac{75}{100} = \frac{3}{4}$, $.665 = \frac{665}{1000} = \frac{133}{200}$.

Exercise 15

Change to common fractions:

1. .5	**4.** .125	**7.** 7.875
2. .65	**5.** .365	**8.** 6.6666
3. .25	**6.** 2.555	**9.** 471.104

REDUCTION OF COMMON FRACTIONS TO DECIMALS

39. To reduce a common fraction to a decimal fraction multiply both the numerator and the denominator by some integer that will make the denominator a power of 10. The easiest way to accomplish this is to multiply both terms of the fraction by a convenient power of 10, then divide both terms by the original denominator. This will leave the denominator a power of 10 and, if the numerator is integral, the fraction is reduced to a decimal.

For example: $\frac{3}{4} = \frac{300}{400} = \frac{75}{100} = .75.$

If, however, we multiply both terms by 10 and then divide by 4 we obtain a fractional numerator, as $\frac{3}{4} = \frac{30}{40} = \frac{7\frac{1}{2}}{10}.$

A fractional numerator can usually be avoided by choosing a larger power of 10.

There are some fractions, however, which cannot be reduced to decimals exactly, no matter how large a power of 10 is taken.

For example: $\frac{1}{3} = \frac{10}{30} = \frac{3\frac{1}{3}}{10}$ or $\frac{1}{3} = \frac{100}{300} = \frac{33\frac{1}{3}}{100}$

In such cases the fraction is called a repeating decimal. The fraction $\frac{1}{3}$ is written .333+ or $.333\frac{1}{3}$. The process is accomplished in practice by the following rule:

40. Rule. *To reduce a common fraction to a decimal fraction annex zeros to the numerator and divide by the denominator, pointing off as many places as there are annexed zeros.*

Change to decimals: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{3}{8}$, $\frac{1}{16}$, $\frac{6}{125}$, $\frac{9}{64}$, $\frac{19}{250}$, $3\frac{3}{4}$, $46\frac{2}{3}$, $23\frac{1}{13}$, $37\frac{1}{2}$, $\frac{19}{984}$, $\frac{29}{876}$.

Take the answer to each of the above and change it to a common fraction by the rule. This will give further practice in changing decimals to common fractions, and at the same time it furnishes a check for the work.

MULTIPLICATION AND DIVISION OF DECIMALS

41. Two or more decimals can be multiplied by reducing them to common fractions and following the rule of § 19. It is better, however, to multiply the numerators and to account for the absent denominators by pointing off as many places as there are zeros in both denominators.

For example: $.31 \times .6 = \frac{31}{100} \times \frac{6}{10} = \frac{186}{1000} = .186$

$4.72 \times .02 = \frac{472}{100} \times \frac{2}{100} = \frac{944}{10000} = .0944$

The abbreviated process is $.31 \times .6 = .186$, and $4.72 \times .02 = .0944$.

42. Rule. *To multiply two or more decimals proceed as in multiplication of integers and point off as many places as there are in both multiplier and multiplicand.*

Division of decimals can also be done by reducing the decimals to common fractions.

For example:

$$.0944 \div .02 = \frac{944}{10000} \div \frac{2}{100} = \frac{944}{10000} \times \frac{100}{2} = \frac{472}{100} = 4.72.$$

The process can be much shortened by dividing as in integers and placing the decimal point by the following rule:

43. Rule. *To divide one decimal by another proceed as with integers and point off as many places as there are in the dividend less the number of places in the divisor. If necessary, zeros may be added to the right of the dividend, these zeros to be counted as decimal places in the dividend when placing the decimal point in the quotient.*

Multiply: $.37\times.41$, $.175\times.24$, $31.72\times19.13\times.007$, $874.65\times.0371$, $.305700\times981.6$, $1841.956\times.39$, $.013605\times 9.75446$.

Check each problem above by dividing the product by one factor. If the quotient is the other factor the result is correct.

Divide: $13.425\div1.754$, $.09423\div.0073$, $471.1\div1881.104$, $.1042\div37.4$, $12.753\div3.425$, $.00316\div12.742$, $12.742\div 00316$.

Check each division by multiplying the quotient by the divisor. If the product is the dividend the work is correct.

Exercise 16—Exercises and Problems

1. $\dfrac{1.71\times374}{1.017}$

2. $\dfrac{875.4\times.94173}{64.43\times.0987}$

3. $\dfrac{.0081\times18.743}{.0007\times.0734}$

4. $\dfrac{9.21\times.647\times13.14}{384.74}$

Check each problem by performing the multiplications and divisions in different order.

For example, (1) can be solved by performing the operations in the order in which they are written, or the order can be varied by dividing 1.71 by 1.017 and multiplying the result by 374. This variation of the order gives more practice and furnishes an excellent check on the work.

5. $69.17+345.4-9.741\times.017$.

(Operations of multiplication and division must always be performed before those of addition and subtraction.)

6. $23.714-67.41\times.072+281.6\div9.53$

7. $\dfrac{61.74-54.51\times.3877}{10735-1481}$

8. A collection of dimes and nickels is worth $2.60. There are twice as many nickels as dimes. How many are there of each?

Let d stand for the number of dimes. There are twice as many nickels.

$$2d = \text{the number of nickels}$$
$$10d = \text{the worth of } d \text{ dimes in cents}$$
$$5 \times 2d = 10d = \text{the worth of } 2d \text{ nickels in cents}$$
$$\text{Then } 10d + 10d = 260$$
$$20d = 260$$
$$d = 13$$

There are 13 dimes and 26 nickels. This checks because 13 dimes and 26 nickels amount to $2.60.

9. A collection of pennies and dimes is worth $3.90. There are 3 times as many pennies as dimes. How many of each?

10. A man buys a farm of 60 acres of timber land and 40 acres of cleared land for $2460.50. The cleared land cost twice as much per acre as the timber land. How much did he give per acre for each kind of land?

11. One number is 4 times another and their sum is 10.5. Find both numbers.

12. In a certain year the United States had 52,000 more miles of railway than Europe. Together they had 402,000 miles. Find the mileage of each.

13. The combined horsepower of an L. and N. freight engine, a Southern Railway passenger engine, and an I. C. electric tractor is 10,250. The horsepower of the freight engine is 1800 more than the electric tractor and 1000 less than the passenger engine. What is the horsepower of each?

14. How many square feet in the walls and ceiling of a room 14 ft. by 14 ft. and 10 ft. high, if two windows 6 ft. by 3 ft., one door 7.5 by 4.3 ft., and one grate 6.2 by 4.1 ft. are subtracted?

15. Mr. W. Holmes bought of Roth Coal Company 4.5 tons of Pure White Moss Coal at $3.75, 7.6 tons of nut coal

at \$4.65. Mr. Roth sends a statement to Mr. Holmes as follows:

Knoxville, Tenn., Aug. 10, 1916.

Mr. W. Holmes, Bought of Roth Coal Co.

4.5 tons Pure White Moss	@ \$3.75	16	87
7.6 tons nut	@ \$4.65	35	34
		52	21

Received payment,

Roth Coal Company.

After Mr. Holmes has sent his check for \$52.21 Mr. Roth mails to him the same bill with "Received payment" written as above and signed with the company's official name. This is called a *receipted* bill.

Make out a receipted bill for the following problems, checking each carefully so that no errors may occur.

A multiplication may be checked by reversing the order of the factors.

16. Mr. R. Murray bought of the Columbia Phosphate Company 59.81 tons No. 1 phosphate at \$12.81, 42.68 tons of No. 2 phosphate at \$12.35.

17. Mr. Wm. Talbott bought of Anderson, Dulin and Varnell 56 yd. cambric at 37.2 cents, 37 yd. gingham at 26.5 cents, 29 yd. of calico at 24.5 cents.

18. Mr. Allen West sold to George Cohan 49.5 lb. coffee at 28.33 cents, 88 lb. sugar at 6.33 cents, 6 lb. prunes at 12.5 cents, 10 lb. tea at 47.5 cents.

19. Reduce to a decimal:

$$\frac{4\frac{1}{5}}{18\frac{1}{4}}\times\left(\frac{4}{9}+\frac{1}{3}\right)$$

20. If 389 tons of drinking water are used on a vessel during an Atlantic voyage, find the approximate number of gallons and cubic feet.

21. The University of Tennessee library bought from the Oxford University Press one set (21 volumes) of new Encyclopaedia Britannica at $4.90 per volume, and one set of Kipling's collected works (11 volumes) at $2.45 per volume. Make out a receipted bill in both pounds and dollars.

22. A Springfield wagon bed is 9 ft. long, 3.5 ft. wide, and 26 in. deep. How many bushels dry measure will it hold when level full?

THE METRIC SYSTEM

44. In the most important countries of Continental Europe, the decimal system of notation is used for weights and measures of all kinds in a form very similar to the American system of counting money. This system is known as the *Metric System.* It is based on the unit of linear measure, which is called the *meter* (measure).

The Latin words for 10 (decem), 100 (centum), 1000 (mille) give rise to the prefixes for the names of denominations of measure smaller than the base; while the Greek words for 10 (deka), 100 (hekaton), 1000 (kilioi), 10000 (myrioi) yield the prefixes in the names of denominations greater than the base.

45. The following are the principal tables of the Metric System:

1. TABLE OF LINEAR MEASURE

10 millimeters (mm.) make	1 centimeter (cm.)
10 centimeters	1 decimeter (dm.)
10 decimeters	1 meter (m.) (39.37 inches)
10 meters	1 dekameter (Dm.)
10 dekameters	1 hektometer (Hm.)
10 hektometers	1 kilometer (Km.)
10 kilometers	1 myriameter (Mm.) (6.214 mi.)

2. TABLE OF SURFACE MEASURE

100 sq. millimeters (sq. mm.) make	1 square centimeter (sq. cm.)(.155 sq. in.)
100 sq. centimeters	1 square decimeter (sq. dm.)
100 sq. decimeters	1 square meter (sq. m.) (1.196 sq. yd.)
100 sq. meters	1 square dekameter (sq. Dm.)
100 sq. dekameters	1 square hektometer (sq. Hm.)
100 sq. hektometers	1 square kilometer (sq. Km.)

3. TABLE OF MEASURE FOR VOLUME

1000 cubic millimeters (cu. mm.) make	1 cubic centimeter (cu. cm.) (.061 cu. in.)
1000 cubic centimeters	1 cubic decimeter (cu. dm.)
1000 cubic decimeters	1 cubic meter (cu. m.)

4. TABLE OF MEASURE OF CAPACITY

10 milliliters (ml.) make	1 centiliter (cl.)
10 centiliters	1 deciliter (dl.)
10 deciliters	1 liter (l.) (1.057 quarts)
10 liters	1 dekaliter (Dl.)
10 dekaliters	1 hektoliter (Hl.) (26.42 gal.)
10 hektoliters	1 kiloliter (Kl.)

5. TABLE OF WEIGHTS

10 millograms (mg.) make	1 centigram (cg.)
10 centigrams	1 gram (g.) (equal to 15.43 gr.)
10 grams	1 dekagram (Dg.)
10 dekagrams	1 hektogram (Hg.)
10 hektograms	1 kilogram (Kg.) (equal 2.205 lb.)
10 kilograms	1 myriagram (Mg.)
10 myriagrams	1 quintal (q.)
10 quintals	1 toneau (T.) (equal 1.102 T.)

Exercise 17 — Metric Measure

1. In 3 Dm. 2 m. 5 dm., how many cm.?

3 Dm. = 30 m., to which add 2 m. making 32 m.
32 m. = 320 dm., to which add 5 dm., making 325 dm.
325 dm. = 3250 cm.

Hence there are 3250 cm. in 3 Dm. 2 m. 5 dm.

2. Reduce 355 cm. to m.

Divide by 10 to reduce to dm., which gives 35.5 dm.; divide this by 10 to reduce to m., which gives 3.55 m. Hence, in 355 cm. are 3.55 m., or 3 m. 5 dm. 5 cm. It is evident that simply moving the decimal point will solve such examples as the above. Thus, 3 Dm. 2m. 5 dm. reduced to cm. gives 3250 cm.

3. Change 565 cm. to Hm.

In 1 Hm. are 10000 cm.
Hence in 565 cm. are $\frac{565}{10000}$ Hm. = .0565 Hm.

4. In 3 Hm. how many mm.?

5. Find approximately how many inches are in 2 Dm.

6. In 1 mile there are how many m.?

7. From Knoxville to Memphis is about 450 miles. Give the distance in Km.

8. How many sq. cm. in $4\frac{1}{2}$ sq. m.?

9. Change 1794 sq. Hm. into sq. yd.

10. How many sq. cm. in 310 sq. in.?

11. Reduce 17,463 cu. mm. to cu. dm.

12. How many cu. cm. in 30,500 cu. in.?

13. In 50 Dl. how many quarts?

14. How many gallons in 2.5 Kl.?

15. Change 35.2 Hg. to g.

16. How many pounds in 2.5 Mg.?

17. A merchant buys 2 q. of coffee at 20 cents per pound. What is the cost?

18. How many 5 gr. capsules can be filled from 2.5 Hg. of quinine?

19. Find cost of 1.5 Dl. molasses at 15 cents per quart.

20. How many m. in 39.4 ft.?

21. From Chicago to Louisville is about 260 mi. Find the distance in Mm.

46. Summary of Chapter II. The four fundamental operations on decimal fractions, the metric system, and a further use of abridged methods in the solution of problems are the most important topics of this chapter. The rules are here collected for reference.

To add or subtract numbers containing decimals write them in a column with the decimal point of one directly beneath the decimal point of those above; then add or subtract as with integers. § 34, 36.

To multiply numbers containing decimals proceed as in multiplication of integers and in the result point off as many decimal places as there are in both the multiplier and multiplicand. § 42.

To divide one decimal by another proceed as in division of integers and point off as many decimal places in the quotient as there are in the dividend less those in the divisor. § 43.

The metric system of weights and measures is coming more and more into use. Hence it is increasingly important to learn to use it.

In the problems below, the student is urged to study carefully the examples that employ the abridged notation and to use this notation whenever possible. Its mastery will make subsequent chapters easier.

Exercise 18 — Review Problems

1. A man bought a horse and carriage for $400. The horse cost .6 as much as the carriage. What did each cost?

2. If $x+.4x+1.6x=21$, what is x?

3. A and B have together \$24.32. A has 7 times as much as B. How much has each?

4. Two men start from the same point and run in opposite directions. The first runs .3 as fast as the second. How far has each run when they are 52 meters apart?

5. The sum of the angles of a triangle is 180°. If one angle is 30° and the second is .2 the third, how many degrees in each?

6. Two angles of a triangle are equal and the third is twice their sum. How many degrees in each?

7. A watering trough is 8 ft. long, 14 in. wide, and 12 in. deep. How many gallons of water will it hold? How many liters?

8. A watering trough is 15 cm. deep. It is twice as wide as deep. It holds 100 liters. Find its dimensions.

9. A man bought 54.85 lb. of sugar at 9 cents per lb. He sold $\frac{1}{2}$ of it at $9\frac{1}{2}$ cents a pound, $\frac{1}{3}$ at 10 cents, and the rest at cost. How much did he gain?

10. The sum of the angles of a four sided figure amounts to 360°. If they are all equal, how large is each?

11. A number added to .5 of itself equals 75. What is the number?

12. Two men are 175 meters apart. If they travel toward each other, one at 7 and the other 8 meters per second, in how many seconds will they meet?

13. If $x+1.6x-\frac{3}{5}x=12$, what is x?

14. What will be the cost of 1678 ft. of lumber at \$49.50 per thousand?

15. It will take 1,468,000 bricks for a certain school building. Some of them cost \$14 per thousand and the remainder \$18. How many are there of each kind if the total cost is \$22020?

16. If a man can travel 33.68 kilometers in .8 of a day, how far can he travel in 7.5 days?

17. Reduce $\dfrac{2\frac{3}{4}}{3\frac{1}{4}} \div \dfrac{2\frac{1}{3}}{3\frac{2}{3}} \times \frac{3}{5} +$ .02 to a decimal.

18. A man bought apples at \$3.60 per bbl. He sold .7 of them at \$3.95 per bbl. and the remainder at \$3.80 per bbl. His total gain was \$45.75. How many barrels did he buy?

19. Seven men and three boys were hired for a week. Each man received 3 times as much as each boy. Their total wages amounted to \$72. What did each receive?

20. A man bought .67 of a ton of hay for \$7.50. What is one ton worth?

CHAPTER III

ALGEBRAIC NUMBERS AND SYMBOLS

INTRODUCTION

47. Positive and Negative Numbers. What is the highest temperature that any thermometer you have seen will record? How far below zero will it read?

The answers to these questions are certain numbers but they are numbers of opposite qualities.

Did you ever know a man who had money and was in debt at the same time? This is often the case.

For example, a man owes a grocery bill of \$10.00 and has just drawn his pay of \$40.00.

Such numbers are said to be of opposite *quality*, for one of them has the power to destroy a part of the other. Such numbers are called *positive* and *negative* numbers. The sign $+$ denotes a positive number, while the sign $-$ denotes a negative number.

Thus, $+5°$ means five degrees above zero, while $-5°$ means five degrees below zero.

There are many numbers that are in effect negative. For example, a man's indebtedness, his debits in his bank account, charges against him at a store, temperatures below zero, time intervals B. C., etc.

Consider the following problem: A man owns a farm worth \$3000.00, horses worth \$1000.00, cattle worth \$500.00, and farm implements worth \$500.00, and he has \$600.00 in the bank. He owes another bank \$2800.00, is indebted to stores \$2000.00, and has other indebtedness to the amount of \$1500.00. What is his financial condition?

The Positive Possessions (property) are:	
Farm	$3000.00
Horses	1000.00
Cattle	500.00
Implements	500.00
Cash	600.00
Total	$5600.00

The Negative Possessions (debts) are:	
Owed to bank	$2800.00
Owed to stores	2000.00
Other debts	1500.00
Total	$6300.00

The sum of the two groups is then

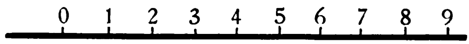
$5600.00 minus $6300.00,
which totals −$700.00.

Hence the man is really worth $700 less than nothing or −$700.00.

SIGNED NUMBERS

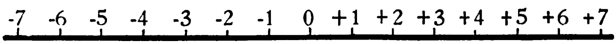
48. Definition. Positive and negative numbers are called *signed numbers.*

The integers of arithmetic may be arranged on a straight line; the distance of each number from the zero represents its size. The number stands on the *end-point* of the line it represents.



Arithmetical Scale

The integers of algebra may be arranged on a straight line extending in both directions from the zero. Thus:



Algebraic Scale

Any point on a straight line can be located if its corresponding number, sign included, is known and if the zero-point is known.

Fractions can be represented by the points between the integers.

49. Addition of Signed Numbers. In arithmetic two numbers are added by starting with one of the numbers and counting forward the number of units in the other, thus, to add 4 and 2, we start with 4 and count forward, 5, 6, that is, two more units.

Two signed numbers are added in the same way except that one must count forward or backward, depending on the *sign* of the numbers to be added.

For example, to add $+3$ and $+7$ begin at 7 to the right of zero and count 3 spaces to the right, to $+10$.

To add -3 to -7 begin at 7 spaces to the left of zero and count 3 spaces to the left, to -10.

To add -7 to $+3$ begin at $+3$ and count 7 spaces to the left, to -4.

These combinations are written $+7+(+3)=+10$, $-7+(-3)=-10$, $+3+(-7)=-4$.

By means of the algebraic scale, § 48, add the following:

$(-3)+(+4)$, $(+3)+(-4)$, $(+7)+(-4)$, $(+5)+(+4)$, $(+5)+(-4)$, $(-3)+(-5)$, $(+9)+(-6)$, $(-6)+(+9)$, $(+6)+(-9)$, $(+7)+(-7)$, $(+2\frac{1}{2})+(-4\frac{1}{2})$, $(-3\frac{1}{2})+(-1\frac{1}{2})$.

The meaning of negative numbers will be made clearer by the following examples:

1. A man gains \$1000 and then loses \$1200. What is the net result?

If gains are represented by positive numbers, losses must be represented by negative numbers. Then

$$(+\$1000)+(-\$1200)=-\$200.$$

2. From morning till 2 P.M. the thermometer rises 28°. From 2 P.M. till midnight it falls 21°. What single change would give the same result? *Ans.* A rise of 7 degrees.

3. A train in 10 hours travels 284 miles east and back 72 miles. How could it have reached the same point by a single direct journey? *Ans.* By traveling 212 miles east.

These examples make clear the following rule:

50. Rule. *To add two numbers with like signs find their sum as in arithmetic and prefix their common sign. To add two numbers with unlike signs find their difference and prefix the sign of the larger.*

The sum of numbers obtained by this rule is known as the *algebraic sum* of the numbers.

51. Subtraction of Signed Numbers. Subtraction is the process of finding a number which, added to a given number, called *subtrahend*, produces another given number, called *minuend.* Thus,

$$7-2=5 \text{ since } 2+5=7.$$

The 5 is the number to be found and we find it by asking what number added to 2 will produce 7.

Signed numbers are subtracted in the same way. Thus, by § 50:

$$(+3)-(-4)=+7, \text{ since } (-4)+(+7)=+3$$
$$(+5)-(+7)=-2, \text{ since } (+7)+(-2)=+5$$
$$(-4)-(-5)=+1, \text{ since } (-5)+(+1)=-4$$
$$(+1)-(-3)=+4, \text{ since } (-3)+(+4)=+1$$
$$(-1)-(-4)=+3, \text{ since } (+3)+(-4)=-1$$
$$(+3)-(+7)=-4, \text{ since } (+7)+(-4)=+3$$

From the above examples it follows that *subtracting* a *positive* number is equivalent to *adding* a *negative* number and *subtracting* a *negative* number is equivalent to *adding* a *positive* number.

For example: $(+3)-(-4)=+7$ is the same as $+3+4=7$
$(+8)-(+4)=+4$ is the same as $+8-4=4$

Hence the rule:

52. Rule. *To subtract one signed number from another change the sign of the subtrahend and proceed as in addition.*

Notice that as soon as negative numbers are admitted, subtraction is always possible. This was not the case before negative numbers

were studied. For example, $5-8$ then had no meaning, but now by § 50, it is equal to -3.

The rules of this and the preceding articles enable us to do away with the double sign in most cases, for instead of $(+5)-(-7)$ we can at once change the sign of the subtrahend and write $+5+7$. Similarly $(+5)+(-7)=+5-7$.

A + sign at the beginning of a series of numbers to be combined is usually omitted. That is, if no sign is written, a + sign is understood. It must be recognized that the signs + and − are used in algebra as symbols of *quality* as well as symbols of *operation*.

For example, $-5°$ means 5 degrees below zero, that is, the minus sign gives a certain quality to the 5°. On the other hand, in $7-5$ the minus sign is a symbol of operation; it says to subtract 5 from 7.

A few occasions arise when confusion will result from omitting the quality sign, but after some experience one can usually tell from the context which use is meant.

Exercise 19

Perform the indicated operations:

1. $12-7$	**8.** -6	**10.** $+41$
2. $11-15$	$+4$	-17
3. $-7+3$	$+3$	-33
4. $-14+16$	$\underline{-10}$	$\underline{+27}$
5. $-4-6$		
6. $21-6$		
7. $+7$	**9.** -7	**11.** -61
-3	$+13$	-43
$+5$	-21	-16
$\underline{-8}$	$\underline{+17}$	$\underline{+125}$

Use rules of §§ 50 and 52, combining all the negative terms, then all the positive, then taking the algebraic sum.

12. The sum of the highest and lowest temperatures on a

certain day was $+12°$. The highest temperature was $+17°$. What was the lowest?

Let t stand for the lowest temperature. Then $t+17°$ stands for the sum of the highest and lowest. But $12°$ stands for this sum; therefore

$$t+17°=12°$$
$$t=12°-17°=-5°$$

Therefore the lowest temperature was $5°$ below zero.

13. The sum of the highest and lowest temperatures on a certain day was $27°$. The highest was $21°$. What was the lowest?

14. The sum of the highest and lowest temperatures on a certain day was $+41°$. The highest was $45°$. What was the lowest?

15. The difference between the highest and the lowest temperatures on a certain day was $+23°$. The lowest was $+10°$. What was the highest?

16. The difference between the highest and lowest temperatures on a certain day was $+19°$. The highest was $+14°$. What was the lowest?

17. A merchant gained an average of $2800 per year for 5 years. The first year he gained $3600, the second $2500, the third $4000, the fourth $4400. Did he gain or lose, and how much, the fifth year?

18. A real estate agent gained $8400 on four deals. On the first he gained $6400, on the second he lost $2100, on the third he gained $5000. Did he gain or lose on the fourth and how much?

ALGEBRAIC SYMBOLS AND LANGUAGE

53. Algebra like arithmetic treats of numbers. In arithmetic, numbers are denoted by combinations of the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In algebra numbers are denoted by these numerals and also by letters.

The rules of arithmetic are often much shortened by the use of letters for numbers. This has already been seen in the chapters on common and decimal fractions. You are doubtless already familiar with the formula:

$$prt = i$$

This is the algebraic way of expressing the law:

The principal multiplied by the rate multiplied by the interval of time gives the interest.

The shortening is remarkable, and it is one of the principal advantages of algebra. It is worth while to notice that the letter p may stand for any principal—\$5, \$20, or \$10,000, as the case may demand. It is in a sense a *general* number. The rules of operation on these general numbers, that is, the rules by which they are added, subtracted, multiplied, and divided must be carefully studied; this subject will be dealt with in the remainder of this chapter.

54. Symbols of Operation. The signs $+$, $-$, $\times$, and $\div$ are used as in arithmetic except that the sign of multiplication is usually omitted. Thus,

principal $\times$ rate $\times$ time is written simply,

$$prt$$

55. Algebraic Number. Any combination of numerals with letters for the purpose of representing numbers is called an *algebraic expression* or an *algebraic number* or simply a *number*.

56. Factors. If a number is the product of two or more numbers any one of them is called a *factor* of the product.

If a number is a factor of two or more numbers it is said to be a *common* factor of those numbers. Thus, in $5x$ and $15y$, the common factor is 5.

57. Coefficient. If a number is the product of two factors either of these two factors is the *coefficient* of the other.

For example, $18r$, $5n$, $8b$, $\frac{3a-x}{c}$ are *number expressions*, or simply *numbers*. In the first, 18 is the coefficient of r, and r is the coefficient of 18. In the second, 5 is the coefficient of n, and n is the coefficient of 5. It is customary, however, to regard the numeral as the coefficient of the literal part.

Since, however, the values of the letters are often not expressed in actual numbers it is necessary to denote operations by means of symbols; thus, the numbers a and b cannot be combined by addition into a single form like 12 as the sum of 7 and 5. Hence, the addition of a and b can only be indicated by writing the numbers in the form, $a+b$.

ADDITION AND SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

58. Addition and Subtraction of Numbers Having a Common Factor. Since 5 horses plus 2 horses equals 7 horses it is clear that

$$5h+2h=7h.$$

How many a's do $3a+4a$ equal? As we shall usually write it,

$$3a+4a=?$$

Again, 7 dollars minus 3 dollars = 4 dollars, or $7d-3d=4d$.

Finally, a drop of 30 degrees in the temperature from 21 degrees above zero is represented by $21d-30d=-9d$.

These examples make clear the following rule:

59. Rule. *To find the sum or difference of two or more numbers having a common factor combine their coefficients by rule* (§ 50) *and multiply the result by the common factor.*

Exercise 20

Find the sum of each of the following:

1. $3-2+7$
2. $3a-2a+7a$
3. $3x-2x+7x$
4. $3\sqrt{2}-2\sqrt{2}+7\sqrt{2}$
5. $3\sqrt{a}-2\sqrt{a}+7\sqrt{a}$
6. $11a-6a+3a+5a$
7. $11ab-6ab+3ab+5ab$
8. $3ab+5ab-6ab+11ab$
9. $11c+5c-19c-3c-25c$
10. $6y+3y-13y+y-16y+3y$
11. $3x-17x+14x+x-2x+3x$
12. $-18xy+3xy-xy+20xy-7xy+10xy$
13. $3\sqrt{x-y}-2\sqrt{x-y}+7\sqrt{x-y}$
14. $-3\sqrt{xy}+7\sqrt{xy}-5\sqrt{xy}$

MULTIPLICATION OF SIGNED NUMBERS

60. Suppose the thermometer is now at zero degrees and is rising at the rate of 4 degrees per hour. What will be the temperature three hours hence at this rate?

The rate at which the temperature is rising is 4 degrees per hour, which we shall consider positive. The time measured in the future we also consider positive. Hence the temperature will be $(+4)\ (+3)=+12$.

Again, suppose the thermometer is now at zero degrees and is falling at the rate of 4 degrees per hour. What will the temperature be three hours hence?

We must now consider the rate of falling as negative since we have considered a rising rate as positive. The time is positive as before. Hence the temperature will be $(-4)\ (+3)=-12$.

Now suppose the thermometer is at zero and has been falling 4 degrees per hour. What was the temperature three hours ago?

The rate of falling is negative. If future time is counted as positive then past time must be considered as negative. Hence the temperature was $(-4)\ (-3) = +12$.

61. Considering carefully the above examples we notice that any other number could have been taken as the rate of rising or falling and any other number as the time. It is clear that the answer to the first type of problem will always be positive, for a rising thermometer will certainly get above zero at some future time. The answer to the second type of problem will always be negative, for a falling thermometer will certainly get below zero at some future time. The answer to the third type of problem will always be positive, for a zero-thermometer which is falling has certainly been above zero in the past. This leads us to the general rule:

62. Rule. *The product of two numbers with like signs is positive. The product of two numbers with unlike signs is negative.*

Exercise 21

Perform the following multiplications:

1. $2(-3)\ 4(-1)$

By the above rule,

$2(-3) = -6,\ (-6)4 = -24,$ and $(-24)(-1) = +24.$

By repeated applications the product of any number of terms can be found. Each negative factor changes the sign of the product. Hence if the number of negative factors is even the result is positive; if the number of negative factors is odd the result is negative. It is better to determine the sign of the product by counting the number of negative terms before the multiplication is performed.

2. $-2(-4)(3)(31)$
3. $8(-2)(16)(-5)$
4. $-50(-12)(-30)$
5. $-7(-5)$
6. $5(-10)$
7. $40(0)$
8. $4(0)$
9. $7(08)(0)$
10. $4(5)(-6)$
11. $4(5)(6)$
12. $8(-3)(-7)$
13. $(-6)(-4)(-2)$

MULTIPLICATION OF ALGEBRAIC NUMBERS

63. We assume that the laws which guided us in working with arithmetical numbers shall also govern our operations with algebraic numbers. A product of several algebraic numbers is generally indicated by simply writing the numbers side by side. That is, ab means a times b. Sometimes, when there are reasons for it, multiplication is indicated by a cross as $a\times b$, $c\times d$ for ab, cd; $u\times v\times w$ for uvw. It is customary in algebra to use a dot, written a little higher than the decimal point, instead of the sign $(\times)$. Thus, $a\cdot b\cdot c=abc$.

64. Order of Factors. In arithmetic $3\times 4=4\times 3$; that is, the order of the factors of a product may be changed. Therefore, in algebra we agree that the factors of a product may be written in any order. This is called the *commutative law* of multiplication.

It is further agreed that the operations of multiplication in the product of several factors may be performed in any order. That is, $a(bc)=(ab)c$. This is called the *associative law* of multiplication. If, in a product, some factor occurs more than once, this fact is indicated by writing a number, called an *exponent*, above and to the right of the factor. *An exponent indicates the number of times a factor is repeated.* Thus,

$$xx=x^2,\ aaa=a^3,\ yyyyyyyyyy=y^{10}$$

By this definition an exponent can be only a positive integer. The definition will later be enlarged to include fractions and negative numbers.

Since $x^3 = xxx$ and $x^2 = xx$, it follows that

$$x^3x^2 = (xxx)(xx) = x^5.$$

Similarly,

$$a \times a^3 \times a^2 = (a)(aaa)(aa) = a^{1+3+2} = a^6$$

Therefore we have the rule:

65. Rule. *To multiply like numbers add their exponents and use the sum as the new exponent.*

For example, by means of the preceding rules and definitions we can now have such multiplications as

$$6a^2b^3(-3ab^2)x^2y$$

for since the order is immaterial the numerical coefficients can be brought to the left hand and combined by the rule of § 62, giving—

$$6(-3)a^2ab^3b^2x^2y = -18a^{2+1}b^{3+2}x^2y = -18a^3b^5x^2y$$

Exercise 22

Perform the indicated multiplications:

$$(4x^2yz)(5xy^3)(-2yz^3) = (4)(5)(-2)x^{2+1}y^{1+3+1}z^{1+3} = -40x^3y^5z^4$$

1. $2(-7)$
2. $2x(-7x)$
3. $2ab^2(-7a^3b)$
4. $(-7c)(2a)$
5. $3(-5a)$
6. $6x^2y(-x^4)$
7. $-6x^4(-x^2y)$
8. $(-7a)(-10)$
9. $4x(-6y)$
10. $3x^3(-2x^2)(-4x)$
11. $(xz)(xz)$
12. $7xy(-2x^2z)(-5yz^2)$

DIVISION OF ALGEBRAIC NUMBERS

66. When fractions were discussed in Chapter I, division of one fraction by another was defined as the process of finding a third number such that if it be multiplied by the second, the product equals the first. It will be advisable to review that section. We define division of one algebraic number by another to be the process of finding a third number such that the product of the second and third equals the first. The word product is used as in the last paragraph, that is, the rules of sign must be observed.

For example: $\frac{6}{3}=2$, since $3\cdot 2=6$

$$-\tfrac{6}{3}=-2, \text{ since } 3\cdot(-2)=-6$$

Division is indicated by the symbol $\div$ or by writing one number above the other. The first number is called the **dividend.** The second is called the **divisor.** The third is called the **quotient.** As in arithmetic $2\div 3$ is written $\frac{2}{3}$, so in division of algebraic numbers $a\div b$ is written $\frac{a}{b}$, and this result can be simplified no further.

$$\text{Similarly, } 2a\div 3b=\frac{2a}{3b}$$

$$\text{and } a^2\div x^2=\frac{a^2}{x^2}$$

$$\text{but } 9x^2\div 3a^2=\frac{9x^2}{3a^2}=\frac{3x^2}{a^2}$$

From the definition of division we find—

$$\frac{-27a}{3}=-9a, \text{ since } 3(-9a)=-27a$$

$$\frac{27a}{-3}=-9a, \text{ since } (-3)(-9a)=27a$$

$$\frac{-27a}{-3}=9a, \text{ since } (-3)(9a)=-27a$$

These examples show that the law of signs in division is exactly the same as in multiplication, that is,

The quotient of two numbers of the same sign is positive, but the quotient of two numbers of opposite signs is negative.

The above examples also show that a factor in the numerator may be canceled by a like factor in the denominator. Thus,

$$\frac{ax^3}{x}=\frac{axxx}{x}=axx \text{ or } ax^2$$

$$\frac{2a^4}{3a^3}=\frac{2a\cancel{a}\cancel{a}\cancel{a}}{3\cancel{a}\cancel{a}\cancel{a}}=\frac{2a}{3}$$

Hence the following rule of exponents:

67. Rule. *To divide like numbers subtract the exponent of the divisor from that of the dividend and use the difference as the exponent of the number in the quotient.*

If the exponent in the dividend is the larger, then the number appears in the numerator of the quotient. Thus,

$$\frac{6a^3x}{2a^2y}=\frac{3ax}{y}$$

If the two exponents are the same in the divisor and dividend then all the factors cancel and the number does not appear at all in the quotient. Thus,

$$\frac{6a^2x}{2a^2y}=\frac{3x}{y}$$

If the exponent in the divisor is the larger, then the number appears in the denominator of the quotient. Thus,

$$\frac{6a^2x}{2a^3y}=\frac{3x}{ay}$$

Exercise 23

Solve as indicated in the first example below.

1. $8 \div -2 = \dfrac{8}{-2} = -4$
2. $-12 \div 6$
3. $8a^3 \div 2a^2$
4. $9a^3 \div 2a$
5. $-9a^3 \div 2a$
6. $6x^6 \div (-2x^2)$
7. $-16y^9 \div (-8y^4)$
8. $-26ax^2 \div 12ax$
9. $\dfrac{-35ay^4}{-5cy^3}$
10. $\dfrac{a^3}{a^5} = \dfrac{\cancel{a}\cancel{a}\cancel{a}}{aa\cancel{a}\cancel{a}\cancel{a}} = \dfrac{1}{aa} = \dfrac{1}{a^2}$
11. $\dfrac{60x^2}{12x^5}$
12. $\dfrac{-75x}{25x^3}$
13. $\dfrac{-84p^9}{-7p^{13}}$
14. $\dfrac{-132a^{11}b^{22}}{11a^{22}b^{11}}$
15. $\dfrac{-18xy^{20}c^9}{54x^3z^{18}c^{15}}$
16. $\dfrac{6x^{2a}}{3x^{2a}}$

68. Summary of Chapter III. Let the student look up and write out carefully the answers to the following directions and questions:

State the rules for addition of numbers with like signs; for addition of numbers with unlike signs; for subtraction of one signed number from another. §§ 50, 52.

Give an example showing how the use of algebraic symbols shortens the statements of rules.

What is an expression? a factor? a coefficient? §§ 55, 56, 57.

State the rule for finding the sum or difference of two or more numbers having a common factor. § 59.

State the rules of sign in multiplication of signed numbers. § 62.

What is the commutative law of multiplication? § 64.

What is the associative law of multiplication? § 64.

Give the definition of an exponent. By this definition can an exponent be a fraction? § 64.

State the laws of exponents in multiplication and division of like numbers. § 65, § 67.

What is the law of sign in division of signed numbers? § 66.

Exercise 24 — Review Problems

Simplify the sums in the first 11 of the following:

1. $xy-8xy+7xy$
2. $(x+a)+19(x+a)-13(x+a)$
3. $x-13a+19a-13x+19x+a$
4. $x-15b+7x-5x-2b+b-3x$
5. $6a^2b+6ab-6ab^2+5a^2b^2$
6. $7a^2x^2y-8a^2xy+13a^2x^2y-3a^2x^2y$
7. $abc-3abc+4abc-3acd$
8. $10ay+z^2-abx-3ay+3z^2+7ay+5abx-8ay-2z^2$
9. $13ab+2cd+25a^2b+3cd$
10. $3bcx-2bcx+3bcx-5bcx$
11. $18a^4xyz+21a^2-3y-3a^2+2y+19a^4xyz-3a^2+24y$
12. Multiply $9xy$ by $6x$
13. Multiply $9ab$ by $-n$
14. Multiply $-a^2$ by $-b^2$
15. Multiply $6a^2$ by $-32a^2b$
16. Multiply $22mn$ by $3p^2q$
17. Multiply $4cn$ by $4x$
18. Multiply $3x^2$ by $-x^2y$

19. Multiply $-2xy$ by $-10zw$

20. Multiply $15x^2y^3$ by $3xy^3$

21. Multiply $21a^3b^2$ by $32ab^2c^3$

22. Multiply $13x^4y^3$ by $4x^5y^4z^2$

23. Multiply $5a^3x^2y^3$ by $-6a^5x^3z^4$

24. Multiply $-6x^3y^3z^3$ by $-4x^3y^3z^3$

25. Multiply together $12a^{11}z^2$, $-5a^2z$, and $6a^3x$

26. Multiply together $5m^2n^2p^2q^3$, $-3m^3n$, and $-7p^5q^8$

27. Multiply together $6ab$, $-2m^2$, $-9ab$, and $3m^2$

28. Divide x^6 by x^2

29. Divide $9x^6$ by $-3x^2$

30. Divide $7x^9$ by x^3

31. Divide $-a^5x^9y$ by $-a^5x^8y$

32. Divide $15x^2y^3z^2$ by $5x^2z$

33. Divide $-10a^3x^2y^2$ by $-2a^2x^2y$

34. Divide $12a^3b^3c^4d^2$ by $-4a^3b^3d^2$

35. Divide $19a^8b^4x^2y$ by $-19abx^2$

36. Divide $-12v^2wn^4x$ by $-15c^6w^2x$

37. Divide $13x^2y^4z$ by $-3a^2x^2y^4z$

Perform the indicated operations:

38. $\dfrac{-24u^5vw^8y^2}{-8uvw^8}+6u^4y^2-16xp$

39. $\dfrac{39f^3g^4h^5k}{13f^3hk}-3g^4h^4+2k^2g^2$

40. $\dfrac{-18x^4ay^3d}{-2xday^2}+\dfrac{24fxga}{-3fx}-17ax^2$

41. Divide \$47 among Harold, Forest, and John so that Harold may have \$1 more than Forest, and John \$3 more than Harold.

42. A boy spends \$24 for a suit and \$10 for books and then has left one third of his money. How much had he?

43. At a certain election 2874 votes were cast. Of the two candidates, one received 376 more than the other. How many did each receive?

Find the value of x in the following:

44. $3x-26=39-8$

45. $.6x=2.4$

46. $2.1x-.7x=.42$

CHAPTER IV

THE EQUATION

69. In the first two chapters we used equations for the solution of certain problems and obtained a limited knowledge of them. In order to be able to use the equation better we were obliged to study the operations with algebraic numbers. The principles developed in the last chapter will now be used in the solution of equations.

DEFINITIONS

70. Algebraic Expression. Any combination of letters and symbols used to represent a number is called an *algebraic expression*, an algebraic number, or simply a number. Thus,

$$\frac{2x+1}{7},\quad a^2-2$$

71. Notion of Equation. An *equation* is a statement of equality of two expressions. Thus,

$$2x+7=9$$

The expressions thus connected are the *members* of the equation. Equations in which letters are used are of two kinds:

1. *Identical equations* or *indentities* like $\frac{6x}{3}=2x$, which are true for every possible value of x.

2. *Conditional equations*, or *equations*, simply, which are true for only one or a few values of the unknown letter and untrue for all others.

For example, $2x=4$ is true if $x=2$, and is untrue for all other values of x.

The values of the unknown which make a conditional equation true are said to *satisfy* the equation. They are called the *roots*, or solutions, of the equation.

To solve an equation in one unknown is to find the value or values of the unknown which satisfy it.

Conditional equations are usually called simply equations. We shall use both identities and equations very often in this work. In order that the idea of an identity may be made as clear as possible a second definition is given.

An *identity* is an equation in which, if the indicated operations are performed, the two numbers become precisely alike, thus,

$$2x(-x) = -2x^2$$

If the indicated operations are performed on the left member, the equality becomes $-2x^2 = -2x^2$, *an identity*.

72. Root of an Equation. It is important also that the ideas of an equation and its *root* be very clear. A certain number is a root of an equation if, when this number is put in place of the unknown, the equation reduces to an arithmetical identity. Thus, in

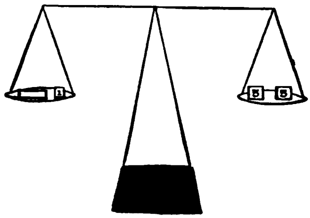
$2x-7=1$, $x=4$ is a root of this equation because if 4 be put in place of x the equation becomes $2\times4-7=1$, which is a true statement.

73. The student doubtless experienced some trouble using the equation in the first two chapters. The unknown quantity probably caused doubt as to how to handle it while the value was yet unknown. Students often ask: "What value has x?" "Is it 3?" "How can I handle it without knowing its value?" To remove this difficulty, let us consider the following problem.

74. Illustration. A man wishes to weigh an iron window weight. He has a pair of balances as shown in the figure on page 63, two 5-lb. weights, and one 1-lb. weight. The weight

of the iron bar is unknown. It is the unknown thing to be determined. The fact that it is unknown, however, need not make it strange or mysterious to us. We can handle it just as we can handle one of the known 5-lb. weights. We place the bar in one pan, and one 5-lb. weight in the other. The bar is the heavier. Trying to make a balance, we find that the 1-lb. weight must be put in the pan with the bar and two 5-lb. weights in the other. It is clear that since there is a balance—

The bar + 1 lb. = 10 lb.



If, instead of two 5-lb. weights, we had ten 1-lb. weights in the right hand pan, we could evidently subtract 1 lb. from each side and still keep a balance, *i.e.*,

The bar = 9 lb.

In order to weigh the bar we must solve the equation

$$w+1=10.$$

An equation, viewed in this light, is nothing but a balance. Anything can be done to an equation which will not destroy the balance. Evidently the same number can be added to both members or subtracted from both members. Then the way to solve the equation

$$w+1=10$$

is to subtract $\quad 1 = \;\, 1$ from both members

and obtain $\quad w \quad = \;\, 9$

Suppose now a different iron bar is to be weighed, and it is found that, in order to make a balance, two of them

must be placed in one pan, and a 5-lb. and a 1-lb. weight in the other pan. This gives the equation

$$2w=6$$

Evidently, there would still be a balance if the weight in each pan were just half as great. According to our hypothesis, however, we have only two 5-lb. weights and one 1-lb. weight. Therefore we are unable in this case to put 3 lb. of weights in the right hand pan, but the value of w can be found by dividing both members of the above equation by 2, giving

$$w=3 \text{ lb.}$$

AXIOMS

75. Common sense tells us that certain facts are true without call for proof. Such facts when made the basis of reasoning are called *axioms*. There are five axioms that are of importance in handling algebraic numbers and equations. The formal statement and arrangement of these axioms are as follows:

Axiom I. If the same quantity or equal quantities be added to equal quantities the sums are equal.

Axiom II. If the same quantity or equal quantities be subtracted from equal quantities the remainders are equal.

Axiom III. If equal quantities be multiplied by the same number or by equal numbers the products are equal.

Axiom IV. If equal quantities be divided by the same number or equal numbers (except 0) the quotients are equal.

Axiom V. The same powers of equal quantities are equal.

From Axiom I it is evident that two or more equations may be added member to member.

From Axiom II it is evident that one equation may be subtracted from another, member from member.

From Axiom III it is evident that two or more equations may be multiplied together and that the resulting products will be equal.

Axiom IV shows that one equation may be divided into another, making the quotients equal.

Axiom V shows that if the two members of an equation are raised to the same power the two powers will be equal.

In general, from the illustrations of the previous section, and from the above axioms, we draw the following:

76. Principle. *The form of an equation may be changed by adding the same number to both members, by subtracting the same number from both members, by multiplying both members by the same number, by dividing both members by the same number (except 0).*

Exercise 25

Solve the following equations and check the results by substituting them in the original equations.

1. Solve for x, the equation $3x-2=1$.

Solution: $3x-2=1$

Adding 2 to both members, $3x-2+2=1+2$

Combining like terms, $3x=3$

Dividing by 3, $x=1$

Check: $3\times1-2=1$

$1=1$

2. $x+4=10$

3. $x-4=6$

4. $2x-7=17$

5. $3x+6=15$

6. $3x=12+x$

7. $5x=4+x$

8. $7x=10+2x$

9. $4a=35-a$

10. $-2y=-3y+27$

11. $2w+1=w+8$

12. $6+2p=p+30$

13. $6x-3=18-x$

14. $3x-15-7x=-23$

15. $-7p+19=25-9p$

16. $-13a-3=-9a-19$

17. $8-6y+10+12y=30$

18. $w+3w+16+2w-w+3=+24$

19. $x-3+5x-11=13+4x-54+5x$

The solutions of the above problems come out integers. They were selected with that in view, because the student needed to concentrate on the process of solving and the check, rather than on the arithmetic necessary to handle more complicated cases. Below are other equations the solutions of which are not all integers.

20. $6x+7=16$

Solution: $6x+7=16$

Subtracting 7 from both members, $6x=9$

Dividing by 6, $x=\frac{9}{6}=\frac{3}{2}=1.5$

Check: $6\,(1.5)+7=9+7=16$

21. $7x-21=148$

22. $2x+4-x+17+3x-41=17$

23. $14x-27+3x-5-16x=2x+13$

24. $275p-151+169p=2417$

25. $17y+30+13y-25y=40$

The solutions of all the above problems are positive numbers. The following problems have solutions that are not all positive and not all integers.

26. $17y+30+13y-25y=0$

27. $2k-17+k-34=-54$

28. $6k+13+2k-6=27-3k-39$

29. $-7k+2-64k=+144$

30. $-2x+1=-4x+3$

31. $-x-17=3x+4$

32. $2w+3=-w+1+2w-5$

The coefficients of all the equations above have been integers. We now attempt to solve some equations with fractions as coefficients.

33. $2x-\frac{1}{2}=\frac{x}{4}+3$

Solution: $2x-\frac{1}{2}=\frac{x}{4}+3$

Multiply by 4, $8x-2=x+12$
Add 2, $8x=x+14$
Subtract x, $7x=14$
Divide by 7, $x=2$
Check: $4-\frac{1}{2}=\frac{2}{4}+3$
$\frac{8}{2}-\frac{1}{2}=\frac{1}{2}+\frac{6}{2}$
$\frac{7}{2}=\frac{7}{2}$

34. $1.5x+2=4+1.25x$

Solution: $1.5x+2=4+1.25x$
Subtract 2, $1.5x=2+1.25x$
Subtract $1.25x$, $.25x=2$
Divide by .25, $x=\frac{2}{.25}=\frac{200}{25}=8$
Check: $(1.5)8+2=4+(1.25)8$
$12+2=4+10.00$
$14=14$

It happens in each of the above examples that though the coefficients are fractions the answer is an integer. This will not always be so. The answer might well be either an integer or a fraction. Do not expect integers in all of the problems below.

35. $\frac{x}{2}-6+x=\frac{1}{2}-\frac{x}{4}$

36. $6.2832n=34.2$

37. $32.2t=274$

38. $\frac{k}{3}+6=-k+\frac{2}{3}$

39. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=7$

BUILDING ALGEBRAIC EXPRESSIONS ON NUMBERS

77. In order to solve problems by means of equations one must be able to translate sentences into algebraic expressions. The expression is merely a shorthand statement of the sentence. For example,

$$2x+3$$

is a short way of writing "a number 3 greater than twice x."

One of the principal advantages to be gained from the study of algebra is that it teaches us to write long sentences in short form. A problem leads generally to an equation. The equation is an abridged statement of the problem.

For example: Four times an unknown number plus 8 equals two times the same unknown plus 16. Find the unknown.

If x represents the unknown, then "four times a certain unknown number" is represented by $4x$. The whole problem then becomes $4x+8=2x+16$. The thing to be carefully noted is that if x be replaced by the expression "an unknown number," the equation becomes the problem. All the problems given in this book can be translated into equations. Students should practice this translation from problem to equation and from equation to problem just as a stenographer practices translation into shorthand and back again.

Exercise 26

Change the following into algebraic expressions:

1. A number 4 greater than x.
2. A number 7 greater than x.
3. A number x greater than 11.
4. A number n greater than x.
5. A number 3 less than b.
6. A number 8 less than b.
7. A number 4 times c.
8. A number 3 less than $a-3b$.
9. A number n greater than $a+2b$.
10. A number n less than 3 times x.
11. One part of 20 is 12. What is the other part?

12. One part of 20 is x. What is the other part?

13. One part of 8 is n. What is the other part?

14. Three times a certain number equals 51. What is the number?

15. Seven times a certain number equals 84. What is the number?

16. A tank holds 1575 gallons. How long will it take a pipe discharging 3 gallons per minute to fill the tank?

The above problems are kept rather easy so that the student needs no definite method to enable him to form his equation or to be sure of his accuracy in its solution. For the more difficult problems which appear in Exercise 27, the following rules will help:

78. Rules. (1) *Read the problem carefully and often enough to be able to write it without looking at the book.*

(2) *Note the unknown and let some letter, perhaps its initial letter, represent it.*

(3) *Translate into an equation by equating two algebraic expressions known to stand for the same quantity.*

(4) *Solve this equation and see whether or not the result will satisfy the original problem.*

Exercise 27

1. The sum of two numbers is 42. Their difference is 12. What are the numbers?

Let n represent the smaller number. Then $n+12$ represents the other and $n+n+12$ = the sum of the numbers,

$$\begin{array}{lrl} \text{or} & 2n+12 & =42 \\ \text{Subtract 12} & 12 & =12 \\ \hline & 2n & =30 \\ \text{Divide by 2,} & n & =15 \\ \text{and} & n+12 & =27 \end{array}$$

Check: These two numbers satisfy the conditions of the problem because their difference is 12 and their sum is 42.

2. The sum of two numbers is 34. Their difference is 18. What are the numbers?

3. The sum of two numbers is 22 and the second is 5 less than the first. What are the numbers?

4. The sum of three numbers is 11. The second is 2 greater than the first, but the third is only 1 greater than the second. Find the numbers.

5. The sum of two consecutive numbers is 19. Find the numbers.

6. The sum of three consecutive numbers is 18. Find the numbers.

7. A 50 ft. lot sold for $560. What was the price per front foot?

8. Dr. Eliot's 5 ft. shelf of books contains 27,000 pages in 50 volumes. What is the average number of pages per volume? The average number of pages per inch?

9. At a family reunion there were 48 people. The number of children was 3 times as great as the number of grown people. How many were there of each?

10. A father is 5 times as old as his son. He was 28 years old when his son was born. How old is the son?

11. Find two consecutive even integers whose sum is 30.

12. It is four times as far from Chattanooga to Jacksonville as it is from Knoxville to Chattanooga. What is the distance from Chattanooga to Jacksonville if the distance from Knoxville to Jacksonville via Chattanooga is 550 miles?

13. A rectangle is 7 inches longer than it is wide. Find its length and breadth if its perimeter is 34 inches.

14. Find 3 consecutive odd integers whose sum is 75.

15. Mount McKinley is 3 times as high as Mount Mitchell. Mount Everett is 8753 feet higher than Mount McKinley. The total heights of all three mountains is 56,002 feet. Find the height of each.

16. A publishing house gained \$8100 by printing 4 books. On the first the gain was \$6300. On the second the loss was \$2100. On the third the gain was \$4500. Was there a gain or a loss on the fourth, and how much?

17. During three years a man gained \$3000. His gain the first year was \$2100, which was three times his gain the last year. What was his gain or loss the second year?

18. Find two numbers whose sum is 2, if double the first equals the negative of the second.

19. Find a number such that its double plus 8 equals 14 plus 4 times the number.

20. Find three consecutive odd numbers whose sum is 3.

21. The sum of two numbers is 11. Their difference is -23. Find the numbers.

Suggestion: Let n represent one number. Then $11-n$ is the other.

22. Find two numbers whose sum is 11 and whose difference is 7.

79. Summary of Chapter IV. In this chapter we have made a careful study of the equation. Let the student review the subject matter and write answers to the following directions and questions:

How many kinds of equations are there? Define each and give examples. § 71.

What is the root of an equation? § 71.

What is meant by saying a certain number satisfies an equation? § 71.

How may we change the form of an equation? § 76.

State the first four axioms about equations. § 75.

How is the solution of an equation checked?

Before attempting the review problems let the student read again the rules in § 78.

Exercise 28—Review Problems and Exercises

1. A farmer has 5 times as many cows as horses. There are 36 animals counting both cows and horses. How many of each does he have?

2. Could you solve problem 1 if the total number of cows and horses were 33 instead of 36?

3. From 64 times a certain number 36 times the same number is subtracted. If the result is 56 what is the number?

4. Could you solve problem 3 if the result were 14 instead of 56? Could you if the result were -42?

5. A pole 30 feet long is stuck up in a pond, one end going into the mud at the bottom 2 feet. The length in the water multiplied by 6 equals the length in the air. How much is in the air?

6. The difference between two numbers is 1 and their sum is 13. What are the numbers?

7. Find r if $6.2832r-3.1416r=27.45$.

8. Find r if $1.06r=.0636$.

9. I met a man driving a flock of geese and said, "Good morning, master, with your hundred geese." He replied, "I have not a hundred but if I had twice as many and four geese more I would have a hundred." How many had he?

10. Solve for x, $2x+1=7$

11. Solve for x, $5x+1=7$

12. Solve for x, $-3x+1=7$

13. Solve for x, $ax+1=7$

14. John bought a certain number of white marbles and 14 times as many red ones. After losing 14 and giving away 30 he had only 16 marbles left. How many white marbles did he buy?

15. There are 3 more counties in Connecticut than in Rhode Island and 6 more in Massachusetts than in Con-

necticut. In all three states there are 2 more than 5 times as many as there are in Rhode Island. How many in each?

16. What is the number that will be doubled by adding 5 to it and subtracting the sum from 41?

17. Solve for b, $1061.06 = b + .06b$

18. Solve for b, $2171.1 = b + .07b$

19. Solve for b, $696.60 = b + .08b$

20. Solve for b, $a = b + .06b$

21. Three pieces of marble weigh 19 lb. more than 15 times the smallest piece. The second weighs 5 times the first and 3 lb. over. The third weighs twice the second. What does each weigh?

22. Divide 58 into three parts so that the first may exceed the second by 9 and the second exceed the third by 17.

CHAPTER V

PERCENTAGE

80. Definition. The term *percent* means by the hundred. It is a contraction of the Latin *per centum,* meaning by the hundred, and hence, *percentage* is the process of computing by hundredths. Thus, 4 hundredths can be written .04, $\frac{4}{100}$, 4 percent, or 4%.

Any percent can be written as a decimal, or as a common fraction.

For example, $6\% = .06 = \frac{6}{100} = \frac{3}{50}$

Exercise 29

Write the following percents as decimal fractions and as common fractions reduced to lowest terms:

1. 3%	**7.** $8\frac{1}{3}\%$	**13.** $33\frac{1}{3}\%$	**19.** 80%
2. 4%	**8.** 10%	**14.** $37\frac{1}{2}\%$	**20.** 125%
3. 5%	**9.** $12\frac{1}{2}\%$	**15.** 40%	**21.** 150%
4. $6\frac{2}{3}\%$	**10.** 15%	**16.** 75%	**22.** 260%
5. $3\frac{1}{3}\%$	**11.** $16\frac{2}{3}\%$	**17.** 50%	**23.** $\frac{1}{2}\%$
6. 8%	**12.** $14\frac{2}{7}\%$	**18.** $66\frac{2}{3}\%$	**24.** $\frac{3}{4}\%$

Check the work by reducing each of the answers back to percents.

For example, 125% reduces to $\frac{5}{4} = \frac{500}{400} = \frac{125}{100} = 1.25 = 125\%$ — a check.

Fifty percent of a number means 50 hundredths of the number, which is equivalent to multiplication by .50. Percentage is therefore only an application of decimal fractions.

25. What is 50% of 100?

50% of 100 = .50 × 100 = 50.00.

26. What is 40% of 100? of 60? of 8? of 2? of $\frac{1}{4}$?

27. What is 10% of 80? of 175? of 122? of 9? of 11? of 10?

28. What is 8% of 275 bushels? of $864.65?

29. What is 6% of 650 sheep? of 2574 tons? of 196 rods?

30. What is $12\frac{1}{2}$% of 375 acres? of 3480 bbl.?

THE ALGEBRAIC EQUATION IN PERCENTAGE

81. In each of the problems 25-30 above, there was a percent, a number of which the percent was to be found, and the answer, which was the result of finding the percent. These three quantities occur in every problem in percentage. They are defined as follows:

82. Definitions. The number of hundredths to be taken is called the *rate* and is denoted by the letter r.

The number of which the percent is to be found is called the *base* and is denoted by the letter b.

The result of finding the percent of the base is called the *percentage* and is denoted by the letter p.

In addition to these three quantities we frequently use the base plus the percentage, which is called the *amount*. Sometimes there is use for the base minus the percentage, which is called the *difference*. The amount is denoted by the letter a, the difference by d.

From the definition of percentage we have the rule:

83. Rule. *To obtain the percentage multiply the base by the rate.*

This rule may be condensed into the equation—

$$\text{(I)} \quad br = p$$

This is an exceedingly important equation. As it is written above it is useful for finding the percentage when the rate and base are given.

Exercise 30 — Examples and Problems

1. What is 60% of 350? A man bought 350 cows and sold 60% of them. How many did he sell?

2. What is $67\frac{1}{2}$% of 900? A man's income is \$900 per year and he spends $67\frac{1}{2}$% of it. What are his annual expenses?

3. How much metal is in 375 tons of ore if the metal is $8\frac{1}{3}$% of the ore?

4. A man invested \$27,000, 18% in bank stock, 8% in bonds and mortgages, 34% in town property, and the remainder in a farm. How much did he invest in the farm?

Let c represent the cost of the farm.

5. An estate is valued at \$250,000. By the will 20% goes to a college, 5% to a public library, and 5% to charity. The remainder is divided equally among 5 children. How much does each get?

Let x represent each child's share.

84. A Derived Algebraic Equation. If we divide both members of Equation I, § 83, by b we obtain—

$$\text{(II)} \quad r = p \div b = \frac{p}{b}$$

which is an algebraic way of stating the following rule:

85. Rule. *To obtain the rate divide the percentage by the base.*

Equation II is useful for finding the rate when the percentage and base are given.

Exercise 31 — Exercises and Problems

1. 10 is what percent of 20?

10 is the percentage, 20 is the base. Hence using Equation II

$$r = \frac{p}{b} = \frac{10}{20} \times \frac{5}{5} = \frac{50}{100} = 50\%$$

2. 12 is what percent of 36? of 30? of 48? of 100?

3. 16 is what percent of 32? of 64? of 8? of 10? of 100? of 1600?

4. $\frac{4}{5}$ is what percent of 1? of 4? of 100? of $\frac{1}{2}$? of $\frac{3}{4}$? of $\frac{4}{5}$?

5. A farmer had forty horses and sold 15. What percent was sold? What percent was left?

6. In a test at a certain university one student spelled correctly 318 words out of 350. What percent was spelled correctly? Incorrectly?

7. A grocer had on hand 900 lb. of coffee. He sold $\frac{1}{3}$ of it one day and $\frac{1}{2}$ of the remainder next day. What percent of the whole was sold? What percent was left?

8. A tenant farmer in central Illinois pays $15 annual rent per acre on land worth $250. Neglecting taxes and other expenses, what percent does the owner realize?

9. If in problem 8 the taxes and upkeep are $5 per acre, what percent does the owner realize?

86. Another Derived Equation. If we divide both members of Equation I, § 83, by r we obtain

$$\text{(III)} \quad b = p \div r = \frac{p}{r}$$

which is equivalent to the following rule:

87. Rule. *To obtain the base divide the percentage by the rate.*

The Equations I, II, and III are much more compact than their corresponding rules. The student is advised to use these formulas in his work. Formula I should be memorized. The other two need no special effort of memory because they are so easily derived from I.

How would you derive Formula II from Formula I?

How would you derive Formula III from Formula I?

Exercise 32 — Exercises and Problems

1. If 20% of a number is 10, what is the number?

20% is the rate and 10 is the percentage.

Hence, using Formula III

$$b=\frac{10}{.20}=\frac{\frac{10}{20}}{100}=10\times\frac{100}{20}=50$$

2. If 20% of a number is 20, what is the number?
3. If $33\frac{1}{3}$% of a number is 300, what is the number?
4. If 5% of a number is 45, what is the number?
5. Find the number of which 88 is 22%.
6. Find the number of which 566 is $33\frac{1}{3}$%.
7. Find the number of which .24 is $\frac{1}{3}$%.
8. Find the number of which .63 is $83\frac{1}{3}$%.
9. Find the number of which 18 is $16\frac{2}{3}$%.
10. Find the sum of money of which $608.13 is 6%.
11. Fire destroyed 20% of a stock of goods which was insured for full value. The insurance paid was $8000. What was the valuation of the stock of goods?
12. A farmer sold 684 bushels, which was 75% of the number of bushels of his wheat crop. Find his total wheat crop.
13. If the rental paid for a house is 12% of its value, what is the value of a house renting for $25 per month?
14. A real estate dealer sold $7\frac{1}{7}$% of his property for $560. What is the value of his property?
15. How much must be invested at $4\frac{1}{2}$% to yield $351 per year for each of five children?

88. Algebraic Equations Involving the Amount. The amount was defined to be the base plus the percentage, or, in the shorter notation of algebra,

$$a=b+p$$

But p is equal to rb; therefore

$$\text{(IV)}\quad a=b+rb$$

This is the formula for finding the amount when the base and rate are given. Subtract b from both members of IV, then divide by b, and we get

$$\text{(V)} \quad \frac{a-b}{b}=r$$

This is the formula for finding the rate when the base and the amount are given.

89. The Parenthesis. The expression $b+rb$ is often written $b(1+r)$. The pair of curved lines about $1+r$ is called **a parenthesis.** Inclosing a quantity in a parenthesis means that we regard the whole expression as one number. In order to remove a parenthesis one must multiply all the terms in it by the number outside. Thus $b(1+r)=b+br$. With this understanding we can write $a=b(1+r)$.

Dividing both numbers by $(1+r)$ gives

$$\text{(VI)} \quad \frac{a}{(1+r)}=b, \text{ or } \frac{a}{1+r}=b$$

The fraction-line acts as a parenthesis upon the denominator, $1+r$.

Equation VI is useful for finding the base when the amount and rate are given.

Exercise 33 — Exercises and Problems

1. A clerk's salary is \$120 a month. He gets a raise of 15%. What is his new salary?

\$120 is the base and 15% is the rate. The amount is desired. Hence by Formula IV

$$a=\$120+.15\times\$120=\$120+\$18.00=\$138.$$

2. A clerk's salary after a raise of 15% is \$184 per month. What was his original salary?

The rate and amount are given, therefore by Formula VI

$$b=\frac{\$184}{1.15}=\$160.$$

NOTE: In each of the following problems the student must decide from what is given which formula is to be used.

3. Mr. T. has 480 acres of land, which is 20% more than his brother owns. How much does the brother own?

4. The tobacco produced in Tennessee in 1909 was 65,000,000 lb., which was 30% more than was produced in Pennsylvania. How much did Pennsylvania produce?

5. A boy grew from 54 lb. to 66 lb. in one year. What percent did he gain?

6. A certain bank in Arkansas increased its capital from \$10,000 to \$25,000. What was the percent of increase of the capital?

7. A barn worth \$1100 is insured for 80% of its value. How much does the insurance cost per year at $\frac{3}{4}$% of the sum for which it is insured?

8. At a certain university 70 freshmen out of 200 made less than 30% on a review quiz of high school algebra. These 70 were put in a review section. What percent of the total class were in the review section?

9. It was found (problem 8) that $15\frac{5}{13}$% of those who were not put in the review section failed. All the rest passed. How many of the original 200 finally passed?

90. The *difference* was defined in § 82, and it may be written in equation form, thus

$$\text{(VII)} \quad d=b-br=b(1-r)$$

Let the student read § 88 carefully again and then solve Equation VII for r in terms of b and d. The solution, if correctly done, gives a formula for the rate when the base and difference are given. This formula is used occasionally in later problems and its derivation by the student will serve to fix the solution of an equation more firmly in mind.

Following the scheme of § 89 solve Equation VII for

b in terms of d and r. This will give a formula for the base when the rate and difference are given.

Exercise 34 — Problems

1. A man invested $850 in bank stock. He sold later, losing $50. What percent did he lose?

2. A man sold land at a loss of $43,275, which was 24% of the amount paid. What was the original investment?

3. During a financial depression a clerk's wages were reduced 20%. He then was receiving $64 per month. What were his original wages?

4. At a bankrupt sale the stock of goods brought $8350 at $33\frac{1}{3}\%$ less than the cost. What did the stock cost?

91. General Directions for the Solution of Problems. In the next five sections there are given definitions and problems on the most important applications of percentages. The student is advised to memorize Formulas I, IV, and VII, and to use only these in his solutions. These three equations involve all the quantities that can arise in percentage problems. If two of the quantities involved in them are given the third can always be found in two steps, as follows:

First, substitute directly in I, IV, or VII the values of the two numbers given.

Second, solve the resulting equation for the third quantity.

The student must use his judgment to determine which formula to use. Some illustrative examples will be solved in each case to make these general directions clearer.

COMMERCIAL DISCOUNT

92. Definitions. A sum deducted from the named price of an article is known as *commercial discount*, or simply as

discount. It is usually computed at a certain rate percent on the price.

Example: What is the selling price of an article marked $7.00 if a discount of 15% is allowed?

Here r equals 15% or .15, b equals $7.00.

Then $p = rb = .15 \times \$7.00 = \$1.05$,

and the selling price is $\$7.00 - \$1.05 = \$5.95$.

The sum left after deducting the discount is known as the *net price.* In the above example the net price is $5.95. For the net price the symbol n may be used. Hence as an equation

$$\text{(VIII)} \quad n = b - p = b - rb = b\,(1 - r).$$

See § 90 for similar form.

93. Successive Discounts. Sometimes several discounts are allowed on the same article. In such cases the first discount is taken as above, then the *second discount* is computed on the first *net price,* the third discount on the second net price, etc.

94. Commercial Discounts are generally of three kinds:

1. *Trade discounts* are reductions from the list price due to trade conditions at the time of sale.

2. *Time discounts* are reductions made for payment of the bill within a certain time.

3. *Cash discounts* are reductions made for the immediate payment of a bill of goods sold on time.

95. Illustration. Find the net price of $250 with discounts of 20% and 5%.

The quantity to be found is the difference, the base and rate being given. Hence $n = b - br = 250 - 250 \times .20 = 250 - 50 = 200$.

Therefore $200 is the new base on which the second discount is to be computed. Using the same formula

$$n = 200 - 200 \times .05 = 200 - 10 = 190.$$

The net price is therefore $190.

Exercise 35 — Examples and Problems

Find the net price of the following bills:

1. \$560 with discounts of 20% and 15%.
2. \$480 with discounts of 10% and 8%.
3. \$6741.55 with discounts 30%, 10%, and 2%.
4. A merchant buys a quantity of goods worth \$165.40 on 60 days time with 5% discount if paid immediately. What is the net price if he decides to pay at once?
5. What is the net price of a bill for \$640 with a trade discount of $12\frac{1}{2}$% and cash discount of 3%?
6. Find a single discount equivalent to the successive discounts in examples 1, 2, 3, and 5. Check your work by computing the discount both ways.
7. Is it possible to have a trade discount, time discount, and cash discount on the same bill?
8. A suit marked \$35 is sold for \$24.50. Find the percent discount. What kind of discount is it?
9. A merchant buys two lots of boys' suits; the same number of suits is in each lot. The list price of one lot is \$20 per suit, and of the other \$30. There was a trade discount of 20%. The net price was \$80. How many suits did he buy?

Let s = the number of suits in each lot. Then $20s + 30s$ = cost at list prices.

10. A farmer bought two flocks of sheep. There were twice as many in one flock as in the other. The price for each in the smaller flock was \$6, in the larger \$7. A discount of 5% was allowed for paying cash. The farmer paid \$399. How many sheep were in each flock?
11. A ball club bought 2 dozen balls. One half of them cost \$18. The other dozen was listed at \$15 but there was a certain trade discount which made the two dozen cost \$27. What was the discount rate?

12. Business houses print on their billheads some such expression as:

"Terms: 60 days net; 2% off for 10 days," or
"Terms: 90 days net; 10 days 5%; cash 6%."

What do these expressions mean?

13. A merchant buys shoes at $31.00 per dozen. He sells half of them for $5 a pair and the rest on sale prices at $3.50 per pair. If the trade discount to the merchant was 6%, what percent profit did he make?

14. Solve Equation VIII for r; for b.

TAXES

96. Definitions. A **tax** is money paid for the purpose of bearing the expenses of government. Taxes are levied by towns, cities, townships, counties, states, and the national government. The taxes levied are of two kinds, poll tax and property tax.

A **poll tax** is levied on male citizens over 21 years old without regard to the property they own. It is not levied in all states.

Real estate is fixed property such as lands, lots, buildings, etc.

Personal property is any movable property, such as money, stocks, furniture.

An **assessor** is an officer appointed or elected to estimate the value of property. He makes up a list of the taxable citizens and the amount of the tax of each.

A **collector** is appointed or elected to collect the taxes.

97. The **rate of taxation** is a certain number of mills on each dollar of assessed valuation or it may be expressed as a decimal. For example, a rate of 1.45 per hundred means

that \$1.45 must be paid on each hundred dollars of assessed valuation. In this book the rate will be expressed decimally, thus .005 will mean 5 mills, or $\frac{1}{2}$ cent, on the dollar.

From this definition of the rate we have the formula

$$\text{(IX)} \quad t = vr$$

which is equivalent to the statement, "The tax equals the valuation times the rate."

Exercise 36

Find the taxes on an assessed valuation of:

1. \$2200 at the rate .0035.
2. \$3400 at the rate .012.
3. \$2100 at the rate .0145.

Sometimes public officers and private citizens wish to know what the rate must be to produce a given tax from a given assessed valuation.

4. Solve Equation IX for r in terms of t and v.
5. What is the rate if the valuation is \$5000 and the taxes \$45?
6. Find r if $v = \$23{,}000$ and $t = \$330$.
7. Find r if $v = \$950$ and $t = \$6.65$.

The income of a city can be changed by changing either the valuation or the rate. For political reasons it may not be wise to change the rate. Let the pupil explain why.

8. Solve Equation IX for v in terms of t and r.
9. Find the valuation if the taxes are \$51, rate .015.
10. Find v if $t = \$160$, and $r = .006$.
11. How much must a city increase its valuation to increase the income \$5150, the rate remaining .0145?
12. To facilitate computation of taxes, assessors usually prepare a table as on page 86. Fill in the incomplete part and use it as much as possible in the remaining problems.

Tax table, rate .015.

Property	Tax	Property	Tax	Property	Tax	Property	Tax
$1.......	.015	$10.....	....	$100....	....	$1000...	....
2.......	.030	20.....	....	200....	....	2000...	....
3.......	.045	30.....	....	300....	....	3000...	....
4.......	.060	40.....	....	400....	....	4000...	....
5.......	.075	50.....	....	500....	....	5000...	....
6.......	.090	60.....	....	600....	....	6000...	....
7.......	.105	70.....	....	700....	....	7000...	....
8.......	.120	80.....	....	800....	....	8000...	....
9.......	.135	90.....	....	900....	....	9000...	....

13. How much tax does a farmer pay who owns 370 acres of land valued at $60 per acre and assessed at 60% of its value, with personal property assessed at $2000 and a two dollar poll tax, the rate being .0085?

14. The real estate of a city is valued at $18,000,000; the personal property is valued at $7,000,000. The rate is .0145. The city sells bonds for $150,000 to buy and improve a public park. The interest on these bonds is $9000 per year. How much must the tax rate be increased to pay this interest and add $10,000 per year to a sinking fund to pay off the bonds?

15. A tax collector's report shows that during a certain year he collected $67,481.20 before October 1. A discount of 5% was given each person paying on or before that date. How much did the collector receive for collecting the tax at 2% commission? There are two ways to compute this commission.

COMMISSION

98. Definitions. When a trader sells property of any kind for another person he is usually paid for his services a percent of the price received, or if he buys for another he receives a percent of the cost price. This percentage is called a *commission* or *brokerage* and the person who does this kind of trading is known as a *commission merchant*, or *agent*, or *broker*. The person for whom the buying and selling is done is called the *principal*. After a sale by an agent or broker, the sum sent to the principal after the commission has been deducted and after all other expenses connected with the transaction have been paid, is called the *net proceeds*, or simply the *proceeds* of the sale.

99. The buying or selling price is the *base*, the percent of commission the *rate*, the commission is the *percentage*, while the remainder after deducting the commission and other expenses from the selling price is the *proceeds*.

100. The commission is, then, the rate times the buying or selling price, and the net proceeds is the selling price less the commission and other expenses.

In equation form these become

$$p = rb$$

$$n = b - rb - e \text{ (where } e \text{ stands for other expenses)}$$

101. Illustrations. 1. An agent sold 500 bales of cotton at an average price of \$50 per bale on commission of 2%. The expense of handling the cotton was 75 cents per bale. What amount was sent to the principal?

500 bales @ \$50 = \$25,000, selling price.

(Commission) $p = rb = .02 \times \$25{,}000 = \$500.00$.

Other expenses 500×75 cents = \$375.00

(Proceeds) $n = b - (rb + \text{other expenses}) = \$25{,}000 - (\$500 + \$375) = \$24{,}125.00$.

Hence, the sum sent the principal was \$24,125.00.

2. An agent buys goods for \$2500 at a commission of 3%. He pays for packing and hauling \$36.00. What is the total cost to the principal?

Here $b=\$2500.00$, $r=3\%$, other expenses $=\$36.00$.

Then (commission) $p=rb=.03\times\$2500.00=\$75.00$.

Total cost $=b+\$75.00+$other expenses $=\$2500.00+\$75.00+\$36.00$ $=\$2611.00$.

Hence, the total cost to the principal was \$2611.00.

Exercise 37 — Examples and Problems

1. What are the commission and net proceeds on the sale of 300 boxes of apples at \$3.00 per box at 2% commission, when handling costs an average of 15 cents per box?

2. A merchant paid an agent 3% commission for buying 1000 bags of coffee of 50 lb. each at 15 cents per pound. The cost of handling the coffee was 10 cents per bag and the freight was 6 cents per bag. What was the entire cost of the coffee?

3. A farmer sends to a commission merchant 50 bales of cotton, which is sold on a commission of 2% at \$60.00 per bale. The same merchant then buys for the farmer 10 mules at \$150.00 each on 3% commission for buying. If the cost of handling the cotton is 15 cents per bale, what sum is due the farmer after paying for the mules and all expenses?

4. If an agent charges $3\frac{1}{4}\%$ commission and receives \$750.00 for handling certain business, what is the amount of the transaction?

5. A clerk receives a salary of \$1200.00 and $1\frac{1}{2}\%$ of his sales as commission. If he receives a total of \$1500.00 what is the amount of his sales for the year?

102. Summary of Percentage. There are three fundamental equations which are studied in percentage,

1. $p=br$, 2. $a=b(1+r)$, 3. $d=b(1-r)$.

Every problem in percentage can be handled by means of them. If two of the quantities are given the third can always be found as follows:

First, choose that equation which has only one unknown quantity after the known quantities have been substituted.

Second, solve that equation for the unknown.

The student must determine for himself which of the numbers given correspond to the letters of the formulas.

He is master of the whole subject of percentage who knows how to use Equations 1, 2, and 3 properly.

Exercise 38 — Review Problems

In each problem select the formula to be used, substitute the known quantities and solve for the unknown.

1. Find the base if $r = 2\frac{1}{2}\%$ and $p = \$106.25$.

2. Find the selling price of goods if a loss of $2\frac{1}{2}\%$ amounts to \$106.25.

3. Find the net price of a bill of goods for \$5800 after discounts of 30%, 20%, and 5%.

4. A house rents for \$270 a year. The taxes, insurance, and upkeep amount to \$80 a year. What percent does the owner realize if the house cost him \$2250?

5. In problem 4 suppose there is an 8% mortgage for \$1400 on the house which the owner assumed when he bought it. What percent does he realize on the \$850 actually invested?

6. In problem 5 suppose the owner secured a 6% second mortgage for \$600. What percent does he realize on the \$250 actually invested? Is this condition likely to occur in business?

7. After a battle 70% of a regiment or 644 men were left. What was the original number of men in the regiment?

8. The land area of the District of Columbia is 60 sq. mi., which is 500% more than the water area. What is the water area?

9. I sold two lots for $600 each. On one I gained 25% and on the other I lost 25%. Did I gain or lose and how much?

10. A town's population increased 20% from 1890 to 1900. It lost 20% from 1900 to 1910. How did its 1910 population compare with that of 1890?

11. A town's population increased 20% from 1890 to 1900. It lost during the next ten years so that the population in 1910 was the same as in 1890. What percent did it lose?

12. A bankrupt can pay only 80 cents on the dollar. What will be the gain or loss percent to a dealer who sold him a wagon at 30% profit?

13. An agent sold 12 pneumatic drills for $695.00 each. He received 20% commission but had to pay the freight and his own expenses which amounted to $737.50. What were his profits?

14. A school district votes a tax of 7 mills. If the teacher receives $80 per month and the property valuation is $94,000 how many months may the school run, assuming that all the tax is used for the teacher's salary?

• 15. What is the difference between a single discount of 40% and two separate discounts of 25% and 15%?

16. On a certain railroad the receipts from passenger travel in 1919 were $1,000,000. Passenger rates were increased 40% in 1920. Travel decreased 20%. The increased cost of operation was $25,000. Find the increase of income to the railroad.

17. If the selling price of goods is $\frac{7}{6}$ the cost what percent is gained?

18. A county desires to build and equip a county high school for \$36,000. If a special levy of $1\frac{1}{8}\%$ is made and $\frac{1}{8}$ the amount is raised from personal property, what is the value of the real estate in the county?

19. The annual premium for an insurance policy of \$60,000 on a store is \$750. Find the rate.

CHAPTER VI

INTEREST AND ITS APPLICATIONS

INTRODUCTION

103. Definitions. *Interest* is a sum paid for the use of money borrowed.

The sum of money loaned is called the *principal.*

The interest is computed at a certain *rate* percent of the principal for one year. The sum obtained by adding the principal and the interest is known as the *amount.*

The *time* for which the interest is computed is usually calculated in years, months, and days, in which the year is taken as twelve months and each month thirty days, or three hundred and sixty days to the year.

The United States Government and many banks, however, compute interest on the year of three hundred sixty-five days, and the fractions of a year in days. Interest computed in this way is known as *exact interest.*

104. Illustrations. 1. What are the interest and the amount on \$600.00 for 2 years and 5 months at 8% per annum (per year)?

Analysis: The interest on \$600 for 1 year is $.08\times\$600.00$, which is \$48.00.

2 years and 5 months $=\frac{29}{12}$ years.

Hence, interest for 2 years and 5 months $=\frac{29}{12}\times$interest for 1 year or $\frac{29}{12}\times\$48.00=\$116.00$.

The amount is $\$600+\$116=\$716.00$.

Hence, the interest asked for is \$116 00 and the amount is \$716.00.

2. Find exact interest on \$730.00 for one year and 76 days at the rate of 10% per annum.

Analysis: The interest for 1 year on \$730.00 at 10% is $.10 \times \$730.00 = \$73.00$.

The interest for 1 year and 76 days (441 days) is $\frac{441}{365}$ times the interest for 1 year, or

$$\frac{441}{365} \times \$73.00 = \$88.20.$$

Hence, the interest required is \$88.20.

105. Algebraic Language. In examples of this kind, as in Chapter V, algebraic language may be conveniently used.

Represent the principal (the base) by b.

Represent the rate by r (expressed as a decimal).

Represent interest by i.

Represent amount by a.

Represent time by t (expressed in years).

Then, from illustrations 1 and 2 it is evident that interest equals the product of the base, the rate, and the time, or, in algebraic language

$$\text{(I)} \quad i = brt$$

Since the amount is the sum of the base and the interest, we have

$$\text{(II)} \quad a = b + brt = b(1 + rt)$$

Exercise 39

1. What is the interest on \$374.60 for 2 years at 5%?

The \$374.60 is the principal, the rate is .05, and the time is 2.
Therefore $i = brt = \$374.60 \times .05 \times 2 = \$374.60 \times .10 = \$37.46$.

Find the interest and amount on:

2. \$2074.10 at 6% for 1 year.
3. \$399.40 at 8% for 3 years.
4. \$563.75 at 4% for 7 years.
5. \$5125.25 at 7% for 4 years.
6. Find the interest on \$380.50 for 1 year 7 months at 5%.

$$\begin{array}{lr} i \text{ for 1 yr.} = \$380.50 \times .05 & = \$19.025 \\ i \text{ for 6 mo.} = \frac{1}{2} \text{ interest 1 yr.} & = \quad 9.51 \\ i \text{ for 1 mo.} = \frac{1}{6} \text{ interest 6 mo.} & = \quad \underline{1.58} \\ \text{Therefore, } i \text{ for 1 yr. 7 mo.} = \text{sum} & = \$30.115 \end{array}$$

Find the interest on:

7. $537.80 for 6 years 3 months at 6%.

8. $836.70 for 3 years 8 months at 6%.

9. $408.30 for 4 years 5 months at 7%.

10. $625.40 for 2 years 7 months at 5%.

11. By the method of § 84 solve Equation I for t, for b, for r.

12. How long will it take $300.00 at 6% to produce $27.00 interest?

13. How long will it require $600.00 to produce $67.50 interest at 8%?

14. What principal will produce $70.00 in 1 year 3 months at 7%?

15. What principal will produce $155.75 in 2 years 1 month at 5%?

16. At what rate will $400.00 produce $24.00 in 1 year 8 months?

17. At what rate will $730.00 produce $105.85 in 2 years 5 months?

18. By the method of §§ 88 and 89 solve Equation II for b, for t, for r.

19. What principal will amount to $327.00 in 1 year 6 months at 6%?

20. What principal will amount to $495.00 in 2 years 1 month at 8%?

21. How long will it take $665.00 to amount to $800.00 at 8%?

22. At what rate will $500.00 amount to $575.00 in 3 years 4 months?

THE SIX PERCENT METHOD

106. Advantage of the Method. Six percent is probably the most common rate of interest. It is the legal rate in

most states of the Union. A special method is therefore given to compute it. This method is of value because of the ease with which interest on $1 can be found.

The interest on $1 for 1 year	=$0.06
The interest on $1 for 1 month	=$0.005
The interest on $1 for 6 days	=$0.001
The interest on $1 for 1 day	=0.00\frac{1}{6}$

107. Illustration. Find the interest on $457.50 for 4 years 3 months 21 days at 6%.

Interest on $1 for 4 years	=$0.24
Interest on $1 for 3 months	=$0.015
Interest on $1 for 21 days	=$0.0035
Interest on $1 for 4 yr. 3 mo. 21 days	=$0.2585

Interest on $457.50 for 4 yr. 3 mo. 21 days = $457.50 × .2585 = $118.26.

Exercise 40

Find the interest on:

1. $384.70 for 1 year 3 months at 6%.
2. $734.40 for 2 years 5 months 12 days at 6%.
3. $267.80 for 4 years 8 months 15 days at 6%.
4. $570.65 for 2 years 3 months 10 days at 5%.

Interest at any rate can be computed by the 6% method by adding or subtracting a proper amount. For 5% subtract $\frac{1}{6}$ of the interest, for 8% add $\frac{1}{3}$ of the interest.

5. $451.30 for 3 years 11 months 25 days at 8%.
6. $835.50 for 1 year 2 months 18 days at 7%.
7. $530.20 for 30 days at 10%.
8. $610.00 for 3 months at 8%.
9. $1741.85 for 1 year 3 months 10 days at 6%.
10. $744.45 for 3 years 6 months 6 days at 6%.

COMPOUND INTEREST

108. Definitions. Interest on the principal and unpaid interest combined at regular intervals is called compound

interest. Such interest is usually compounded annually, semi-annually, or quarterly. By far the most common way is semi-annually.

Compound interest is not used greatly in the business world of today. Most business concerns that loan money insist that the interest be paid semi-annually, or annually, thus avoiding the necessity for compound interest. Its greatest use is (1) savings bank accounts, (2) in computing the cost of an investment, and (3) by insurance companies and building and loan associations to calculate the income from investments where the interest is immediately reinvested.

109. Method of Computing. Compound interest is usually computed by means of a table. Such a table is given here correct to six decimal places. The student is asked to verify the 3% column.

Compound Interest Table

Yr.	1½%	2%	2½%	3%	3½%	4%
1	1.0150 000	1.0200 0000	1.0250 0000	1.0300 0000	1.0350 0000	1.0400 0000
2	1.0302 250	1.0404 0000	1.0506 2500	1.0609 0000	1.0712.2500	1.0816 0000
3	1.0456 784	1.0612 0800	1.0768 9062	1.0927 2700	1.1087 1787	1.1248 6400
4	1.0613 636	1.0824 3216	1.1038 1289	1.1255 0881	1.1475 2300	1.1698 5856
5	1.0772 840	1.1040 8080	1.1314 0821	1.1592 7407	1.1876 8631	1.2166 5290
6	1.0934 433	1.1261 6242	1.1596 9342	1.1940 5230	1.2292 5533	1.2653 1902
7	1.1098 450	1.1486 8567	1.1886 8575	1.2298 7387	1.2722 7926	1.3159 3178
8	1.1264 926	1.1716 5938	1.2184 0290	1.2667 7008	1.3168 0904	1.3685 6905
9	1.1433 900	1.1950 9257	1.2488 6297	1.3047 7318	1.3628 9735	1.4233 1181
10	1.1605 408	1.2189 9442	1.2800 8454	1.3439 1638	1.4105 9876	1.4802 4428

The compound interest on any amount for 4 years at 8% payable semi-annually is the same as upon the same amount for 8 years at 4% payable annually.

Use of the Table. Find amount of $1200.00 for 5 years at 3% compounded annually.

In the table in column headed Yr. follow down till you find 5, then *go to the* right to the column for 3%. Here find $1.159274 as the com-

pound amount of $1.00 for 5 years at 3%. The amount for $1200.00 is twelve hundred times as much as for $1.00, hence,

$$1200 \times 1.159274 = 1391.13$$

and the desired amount is $1391.13.

Exercise 41

By means of the table compute the amounts of the following:

1. $842.00 for 3 years 2 months at 6%, payable semi-annually.

2. $510.60 for 4 years 8 months at 6%, payable semi-annually.

3. $5420.70 for 11 years 6 months at 5%, payable semi-annually.

4. $3475 for 16 years 8 months at 4%, payable annually.

5. As a check on your work and the accuracy of the table compute each of the above problems without using the table.

6. Compute the simple interest on each of the above problems and compare it with the compound interest.

7. A man invested $5125 in vacant city lots. Money is worth 6% compounded semi-annually. His taxes on the lots are $78 per year. How much has the property cost him at the end of 18 years?

8. A man paid $850 for 7 shares ($100.00 each) of national bank stock. It paid regularly 16% dividends for 5 years. These were invested in a savings bank at 4%. How much did he receive during the whole ten years and what was the average rate percent realized on his investment?

A savings bank pays 4% compounded semi-annually. It will not pay interest on fractions of a dollar. Interest is computed from the first of the month following a deposit unless the deposit is made during the first five days of the month, in which case it is computed from the first of the month in which it is made. The following is an exact copy of two savings accounts in a certain bank. Verify them in detail.

9.

1911, Oct. 12,	*D*....$5.00	Balance....$ 5.00
1912, Feb. 6,	*D*.... 1.00	6.00
June 4,	*D*.... 1.54	7.54
June interest July 1,	*D*.... .14	7.68
Dec. interest Jan. 2,	*D*.... .14	7.82
1913, Jan. 2,	*D*.... 1.00	8.82
June interest July 1,	*D*.... .16	8.98
Dec. 26,	*D*.... 2.00	10.98
1914, Dec. int., Jan. 2,	*D*.... .16	11.14
June interest July 1,	*D*.... .22	11.36
Sept. 23,	*D*.... 2.50	13.86
1915, Dec. int., Jan. 2,	*D*.... .24	14.10
Jan. 12,	*D*.... 1.50	15.60

10.

1911, Oct. 12,	*D*....$5.00	Balance....$ 5.00
1912, Feb. 6,	*D*.... 1.00	6.00
June interest July 1,	*D*.... .12	6.12
Dec. interest Jan. 2,	*D*.... .12	6.24
1913, Jan. 2	*D*.... 1.30	7.54
June interest July 1	*D*.... .14	7.68
Dec. 26	*D*.... 2.00	9.68
1914, Dec. int. Jan. 2,	*D*.... .14	9.82
June interest July 1,	*D*.... .18	10.00
Sept. 23,	*D*.... 1.80	11.80
1915, Dec. int. Jan. 2,	*D*.... .21	12.01

11. A father on the birth of his son deposited $100 in a savings bank to the son's credit at 4%, compounded annually. How much did the boy have at his 21st birthday?

12. Secure some local problems on savings bank deposits from the members of your class or your friends and solve them.

PROMISSORY NOTES

110. Definitions. A written promise to pay a sum of money at a specified time is called a *note*.

The person who signs the note is called the *maker*. The person to whom the note is payable is called the *payee*.

The person who owns the note is called the *holder*. The sum promised is called the *face* of the note. When no rate of interest is specified the legal rate is understood.

111. Forms of Notes. Notes are written in various forms. Below are two examples of the most common forms:

$250.00. Spokane, Wash., Jan. 2, 1913.

Six months after date, I promise to pay to James Speaker, or order, two hundred fifty dollars, for value received.

John S. Edmiston.

$704.00. Birmingham, Ala., Oct. 4, 1914.

For value received, one year after date, I promise to pay to Richard Sisk or bearer seven hundred four dollars with interest at 5%.

Sam Y. Pyeatt.

A note should contain:

1. The place and time when made.
2. Its face in figures and writing. In law, the writing is the face of a note, not the figures.
3. It should promise to pay to a certain person, to his order, or to bearer.
4. It should state when, to whom, by whom, and sometimes where the money is to be paid.
5. It should state for value received. Otherwise the holder may be put to trouble to prove that the maker got "value received."

112. Change of Ownership. A note may be bought and sold (termed *negotiable*) except when it is made payable to a specific person, in which case it is not *negotiable*. The first holder of a note is the *payee*. If he wishes to sell it he may do so by *indorsing* it, that is, writing his name across the back. He may indorse it (1) in blank, (2) in full, (3) without re-

course. Indorsing "in blank" means that he writes his name, nothing else, on the back, thus making the note payable to the holder, and at the same time the indorser becomes responsible for the payment of the note in case the maker fails to pay. Indorsing "in full" makes the note payable to a definite person whose name is written on the back. Indorsing "without recourse" means that the indorser refuses to become responsible for the payment of the note. Business men are seldom willing to buy a note indorsed "without recourse."

113. Maturity. A note is said to be *due* on the date when it is to be paid; this date is called the day of *maturity* of the note.

Notes may not bear interest for a certain time as until maturity. If not paid at maturity such a note bears interest at the legal rate from the date of maturity.

A *joint and several* note is made by two or more people who individually become responsible for its payment.

Exercise 42

1. Write a promissory note due in 3 months for $80.50 in which you are the maker, Smith, Wheelock and Company, the payees, with interest at 6%.

2. Compute the amount of the note above.

3. Smith, Wheelock and Company sell this note, but do not want to be responsible for its payment. What do they write or stamp on the back? What is this sort of indorsement called?

4. Write a joint and several note for $4000 dated January 19, 1915, due in 6 months to Henry Johnston. You and your teacher or father may sign this note as makers. This note is not to bear interest unless it is unpaid at the end of six months.

5. Find the amount which will pay the note in example 4 on February 19, 1916, at the legal rate in your own state.

6. Why is it safest to indorse a note in full if it is to be sent through the mail? Why is it not good policy to make the note to bearer?

From the following data write negotiable notes, observing carefully the things a note should contain. Find the amount due at settlement.

7. Place, Grove, Ark.; date, Feb. 20, 1911; face, $1700.00; time, 3 yr.; payee, James Knox; maker, J. T. Hobson; interest rate, 8%; settled, May 10, 1915.

8. Place, Lincoln, Ill.; date, July 6, 1913; face, $420.35; time, 6 mo.; payee, John Green; maker, H. T. Brooks; no interest; settled, March 1, 1915.

9. Place, Jacksonville, Fla.; date, March 26, 1910; face, $2175.40; time, 2 mo.; payee, Attis & Co.; makers, M. Jones and C. A. Morse; interest rate, 6%; settled, May 25, 1910.

Interest notes bear interest at the legal rate of the state in which they are made from their maturity till settled.

10. Find some local problems that have occurred among your friends and solve them.

PARTIAL PAYMENTS

114. Definitions. When a payment, which does not pay the entire note, principal, and interest, is made on a note, such payment is called a *partial payment.*

A statement that this payment has been made is written on the back of the note with date, and signed by the payee. Such statement is known as an *indorsement* of the note.

115. Method of Computation. There are several methods used in finding the amount due on a note on which partial payments have been made, but the method accepted by the

Supreme Court of the United States is perhaps most accurate as well as most used. The method is as follows:

116. Rule. *Find the interest on the note to the date of the first payment. If the payment exceeds the interest, add the interest to the principal and from the sum subtract the payment. Compute the interest on this sum to the date of the second payment; if this payment is greater than the interest proceed as before, and so on to the final date. If, however, a payment is less than the interest, then compute the amount to the time when the sum of two or more payments equals or exceeds the interest and proceed as above.*

117. Illustration. A note for $1500 at 8% was given June 1, 1907, and settled January 1, 1910. On the back there were three indorsements: August 1, 1907, $200; June 1, 1908, $55; October 1, 1908, $360.

Principal	$1500.00
Interest June 1 to August 1	20.00
Amount August 1	1520.00
Less first payment	200.00
New principal	1320.00
Interest from August 1, 1907, to June 1, 1908, is $88.00, which is larger than the payment. Therefore we compute interest August 1, 1907, to October 1, 1908	123.20
Amount October 1, 1908	1443.20
Sum of two payments	415.00
New principal	1028.20
Interest to January 1, 1910	102.82
Amount required to settle in full	$1131.02

Exercise 43

1. A note for $1000 dated October 19, 1912, and settled January 19, 1915, had the following indorsements: December

19, 1913, $250; November 19, 1914, $400. Find the balance due on the date of settlement if the interest rate is 6%.

2. A five-year note for $600 dated May 11, 1910, bearing 6% interest, had the following indorsements: November 11, 1910, $18; May 11, 1913, $200; May 11, 1914, $200. Find balance due May 11, 1915.

3. A mortgage for $1275 at 8% dated January 28, 1908, had indorsements as follows: November 28, 1908, $175; January 28, 1909, $200; June 28, 1909, $40; September 8, 1909, $160. Find the amount due January 8, 1910.

BANKING PRACTICE

118. Definition. A *bank* is an institution that receives and loans money.

A *check* is an order to the bank to pay a certain sum to another person. A check can be drawn only by someone who has money deposited in the bank on which it is drawn.

119. Benefits from Banks. A business man deposits his surplus in a bank (1) because he believes it a safer place than his home or place of business; (2) because he can pay his debts much more easily by check than by cash; (3) because a check is always returned to him by the bank on which it is drawn and becomes a receipt.

120. Business Practice. A bank is allowed by law to loan its capital and a certain percent of the money deposited with it. The rules that most banks follow in loaning money differ from the practice among private individuals in that:

1. The bank requires interest in advance.
2. The bank computes interest for the exact number of days.
3. The bank loans for a short term, usually not over three or four months.

4. The bank requires security in the form of stocks, bonds, mortgages, or an indorsement.

5. The bank computes interest on the value of the note at its maturity.

Bank discount is simple interest on the value of the note at its maturity for the exact number of days in the term of the note after date of discounting.

121. Borrowing from a Bank. A man in need of ready money may go to a bank and secure it provided the bank has enough money on hand to supply him and provided he furnishes sufficient security. In the following paragraphs, it is assumed that he is borrowing directly from the bank on his own note.

122. Proceeds of a Note. The face value of his note less the bank discount is called the *proceeds* of the note.

For example, the proceeds of a note of \$665 for 90 days at 8% exact interest, is $\$665.00 - \$665 \times .08 \times \frac{90}{365} = \$665 - \$13.12 = \$651.88$.

This definition is equivalent to the statement:

The proceeds of a note = the face value − the discount. But since the discount equals the interest on the face of the note for the indicated time we have the algebraic formula

$$\text{(III)} \quad p = f - frt = f(1 - rt)$$

The student is asked to review § 89 in the chapter on Percentage where the parenthesis ( ) is explained.

Exercise 44

1. Mr. R. E. Brooks borrows \$610.00 for 90 days from the Mechanics Bank and Trust Company. The rate is 8%. What are the proceeds?

2. R. W. Wolfe borrows \$500.00 for 60 days from the Capital City Bank, Madison, Wis. The rate is 6%. Find the proceeds.

3. A farmer feeding cattle needed $1275.00 to meet his expenses while the cattle were fattening. He borrowed $1300 from the First National Bank of Springfield, Ill., for 90 days. At 7% what were his proceeds? How far did they fall short of his actual needs? Find the exact amount he should have borrowed to get exactly $1275.

4. Take Equation III

$$p=f(1-rt)$$

and divide both sides by $(1-rt)$. What formula do you get?

(Review again § 89 in Percentage.) Solve the last part of problem 3 by this formula.

5. A man needs exactly $650. The bank will make him a loan for 60 days at 8%. What must be the face of the note?

6. The mortgage on J. B. Hutton's house is $1400. When the mortgage falls due he has $250 and must borrow the remainder on a 90 day note at 6%. What must be the face of the note he signs at the bank?

7. J. E. Johnston signs a note for $850 for 90 days. He receives credit on his bank account for the proceeds of the note which is $833. What is the rate of discount?

8. Take Equation III

$$p=f-frt$$

and solve it for r (compare § 89). Use the formula to solve problem 7.

9. The proceeds of a 60 day note are $1831.50. The face is $1850. What is the rate?

10. The proceeds of a 30 day note for $2500 are $2483.33. What is the rate?

11. Take Equation III and solve it for t, starting with the formula

$$p=f-frt$$

12. A 6% note for $270 yields proceeds of $265.95. Find the time.

13. A 7% note for \$750 yields proceeds of \$745.62. Find the time.

Assuming that the student has solved problems 3 to 13 inclusive, by the formulas he is asked to derive in 4, 8, and 11, let him check his work by solving them again according to the model solution below.

14. For what sum must I draw my note so that when discounted at 6% for 3 months, I may realize \$2758?

Using Formula III

$$p=f(1-rt)$$

and substituting \$2758 for p, .06 for r, and $\frac{1}{4}$ for t, we have the equation

$$2758=f(1-.06\times\tfrac{1}{4})$$

Simplifying $2758=f(1-\frac{6}{400})=f(\frac{394}{400})$.

Therefore $2758=\frac{197}{200}f$.

Dividing both members by $\frac{197}{200}$ we get $2758\times\frac{200}{197}=f$, or $f=\frac{551600}{197}$ $=2800$. This will give in most cases a check on the problem, but its greatest value will be derived from the practice in handling the equation.

123. Discounting Notes at the Bank. In §§ 120 and 121 we studied bank discount when the individual was borrowing directly from the bank on his own note. Now we study bank discount when a customer sells (or discounts) the note he holds against a third person.

124. Illustration 1. Mr. Southard holds a note for \$200 dated September 1, 1913, due in three months. This note was made by John Candler and bears no interest. Mr. Southard being in need of funds takes the note to the bank on October 2, 1913, and asks them to discount it. The bank, finding that the note is good and has 60 days to run, advances Mr. Southard \$200, less the interest on \$200 for 60 days at 6%. The proceeds of the note are therefore $\$200-\$200\times.06\times\frac{12}{73}=\$200-\$1.99=\$198.01$. At the expiration of the 60 days the bank collects \$200 from Mr. Candler. This process is almost exactly like that in § 122, and needs no further explanation.

125. Illustration 2. If, however, the note of Mr. Candler had been bearing interest at 6% the computation would have been based on the amount of the note at its maturity as follows:

Amount of note $=200+200\times.06\times\frac{1}{4}=200+50\times.06=200+3=203$.

Then proceeds $=203-203\times.06\times\frac{1}{7}\frac{2}{3}=203-2.00=201.00$.

126. Use of Formula. The *proceeds* of an interest-bearing note = the amount of the note at maturity − the bank discount. This is equivalent to the algebraic formula

$$\text{(IV)} \quad p=a-art$$

In the application of this formula one must be careful to compute a from the rate mentioned on the face of the note and for the term of the note, while the t which appears in the formula is the time from the date of discounting to the date of maturity.

127. Illustration. Find the proceeds of a 90 day note for \$1140 dated May 4, 1914, with interest at 6% and discounted June 18, 1914, at 8%. The amount of the note at maturity = $\$1140+\$1140\times.06\times\frac{1}{4}=\$1140+\$68.40\times\frac{1}{4}=$ $\$1140+\$17.10=\$1157.10$.

The note has 45 days or $\frac{1}{8}$ of a year to run when it is discounted. Hence, Proceeds $=a-art=\$1157.10-\$1157.10\times.08\times\frac{1}{8}=\$1157.10-\$1157.10\times.01=\$1157.10-\$11.57=\$1145.53$.

Exercise 45

1. A note for \$900 bearing interest for 3 months at 6% was dated January 15 and discounted February 20 at 6%. Find the proceeds.

2. $764.75 Opelika, Ala., October 4, 1914.

Three months after date I promise to pay to Albert Heber or order seven hundred sixty-four $\frac{75}{100}$ dollars, value received, with interest at 6%.

J. B. Barton.

Discounted November 10, 1914, at 8%. Find the proceeds.

3. $800.54 Montgomery, Ala., April 8, 1913.

One year after date I promise to pay to A. K. Dustan or order eight hundred $\frac{54}{100}$ dollars with interest at 8%, value received.

E. T. Froman.

Discounted May 5, 1913, at 10%. Find the proceeds.

4. $438.22 Jackson, Tenn., December 6, 1914.

Four months after date I promise to pay to the order of Cary Allen four hundred thirty-eight $\frac{22}{100}$ dollars with interest at 7%. Value received.

Edmund Spencer.

Discounted February 4, 1915, at 8%. Find the proceeds.

5. The proceeds of a note, when discounted at 6% sixty days before maturity, were $591. What was the amount of the note at maturity? What was its face if it was a 90 day note at 6%?

128. Interest Tables. Most bankers use interest tables to compute their interest. Such a table gives the interest on $1, $2, etc., up to $10 for convenient intervals of time, varying from 1 day to 120 days.

129. Summary of Chapter VI. The whole of this chapter is built around four fundamental equations,

1. $I=brt$
2. $a=b(1+rt)$
3. $p=f(1-rt)$
4. $p=a-art$

The first is an abbreviation for the law: Interest equals base multiplied by the rate multiplied by the time.

The second states that the amount equals the base plus the interest.

The third gives the proceeds of a note as its face less its discount.

The fourth gives the proceeds of a note on which interest has accumulated as the amount minus the discount.

All the problems of this chapter may be solved by using some one of these equations. The student must be able to decide which formula can be used. He should choose that equation in which there is only one unknown quantity, and solve for this unknown.

Exercise 46—Review Problems

1. The government of France paid interest at the rate of 4% per annum on a warrant 80 days overdue. The amount of the warrant is $650,000. Find the interest paid.

2. What does the formula $i=brt$ mean? Solve this equation for b; for r; for t.

3. A note for $500, interest 5%, amounted to $563.75 when settled. How long had it been bearing interest?

4. Write the formula for the amount in terms of base, rate, and time. Solve for b, for r, and for t.

5. In what time will $180 amount to $225 at 5%?

6. What principal for 3 mo. at 8% will yield the same interest as $5100 for 5 yr. 6 mo. at 6%?

7. A man bought a tract of land for $5000 paying $1000 cash and borrowing the rest at 6% in four notes of $1000 payable in 1, 2, 3, and 4 years, respectively. How much interest does he pay in all?

8. A invests \$1000 at 6% and B invests \$5000 at 4%. In how many years will A's investment, principal, and interest equal the interest on B's investment?

9. A invests \$600 at 5% and B invests \$1000 at 5%. In how many years will A's interest differ by \$350 from B's interest?

10. How much must I send my broker so that he may buy \$1500 worth of bonds and have 5% commission?

11. What is the formula by which the proceeds of a note are found in case one is borrowing directly from a bank? Solve this equation for f; for r; for t.

12. What is the formula for computing the proceeds of a note held by one individual against another when the holder of the note discounts it at the bank?

13. Solve the formula of problem 12 for a; for r; for t.

14. For what sum must I give my note, discounted for 60 days at 6%, in order to realize \$1025?

15. For what sum must I give my note, discounted for 90 days at 8%, in order to realize \$2758?

16. I hold a note due in one year against S. P. Erwin for \$750, dated July 11, 1914, interest 6%. If I take this note to the bank January 2, 1915, and discount it at 8%, what will be the proceeds?

17. What should a bank pay for a note of \$1200 bearing 8% interest, dated April 10, due in 4 months, if purchased May 1 at 7% discount?

18. A broker buys a \$300 note (without interest) thirty days before maturity for \$297. Find the rate of discount.

19. The amount of a note at maturity will be \$400. It is discounted 60 days before maturity. The discount is \$6.00. *Find the rate.*

20. A man's bank account is overdrawn $269.30. He presents May 10 two notes to be discounted and the proceeds placed to his credit. Find his balance if the notes are as follows:

Face	*Date*	*Time*	*Rate*
$280	March 10	5 months	Without interest
$375	April 1	3 months	8%

CHAPTER VII

MENSURATION

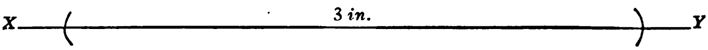
LINES AND ANGLES

130. Introduction. In order that the areas which are much used in everyday life may be thoroughly understood it is necessary that their component parts, the lines and angles which form them, be studied. To do the work of this chapter the student should have a ruler marked in inches (straight edge) and a pair of dividers (compasses) for drawing circles.

131. The Straight Line. Our idea of a straight line comes from the experience that we "see in a straight line." That this is so, is evident from the fact that we sight along a ruler to see if it is straight, that is, we compare it with our line of sight. No clearer idea of a straight line than this can be given.

In making all the drawings keep the pencil sharp and make the lines just heavy enough to be seen clearly.

To draw a straight line 3 inches long the most accurate way is to draw a straight line, XY, put the compasses on the ruler so that the points are 3 inches apart, then place both points of the compasses on the line XY. This gives a line which is 3 inches long more accurately than can be made with the ruler alone.

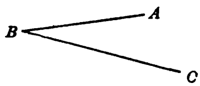


The student is expected to use the above method in the following pages.

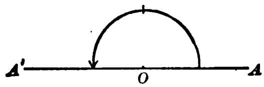
Conversely, to measure the length of any straight line,

spread the dividers so that one tip touches each end of the line, transfer the divider tips to the ruler, and read off the length from tip to tip.

132. Definitions. An *angle* is the opening between two straight lines. Thus the lines AB and BC meet at B, forming the angle ABC. The point B is called the *vertex*, the lines AB and BC are the *sides*. Draw an angle with sides 2 inches long. Extend the sides of this angle till they are 4 inches long. Is the new angle any larger? Does the angle depend on the length of its sides?



133. If the line OA be turned about the point O as indicated by the arrow until $A'OA$ is a straight line, then the angle $A'OA$ is called a *straight angle.*



In the measurement of angles the straight angle is divided into 180 equal parts and each part is called a **degree.**

A *right angle* is half a straight angle. It has 90 degrees in it.

An *acute angle* is less than a right angle.

An *obtuse angle* is greater than a right angle.

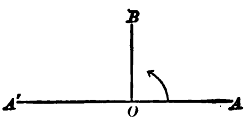


Fig. 1
$A'OB = AOB =$
right angle

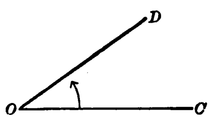


Fig. 2
$DOC =$
acute angle

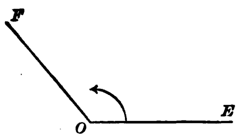


Fig. 3
$FOE =$
obtuse angle

Exercise 47

1. If a line OA 3 inches long be turned about O till it comes back to its original position, how many degrees has it turned through?

2. In problem 1 if the line is 2 inches long, would the answer be the same?

3. Make a line $3\frac{1}{4}$ inches long.

4. How long is this line A ——————————— B ?

5. Two towns are $4\frac{1}{2}$ miles apart. Can their positions be represented by a line $4\frac{1}{2}$ inches long? Draw such a line as carefully as possible.

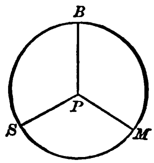
6. A third town is 3 miles and $5\frac{1}{4}$ miles, respectively, from the two mentioned in problem 5. Draw a figure representing their distances. Put the numbers on the lines to which they correspond.

This is called *drawing to scale*. The scale in problems 5 and 6 is 1 inch for 1 mile. Any scale may be chosen so that the figure may be drawn of any convenient size.

CONSTRUCTION OF ANGLES

134. The accurate drawing of a figure is called a *construction*.

135. I. *To construct a circle.* Place one point of the compasses upon a point P with some convenient opening between



the points of the compasses, say PM. Keep the point at P fixed and with the opening of the compasses unchanged turn the other leg of the compasses about the point P.

136. Definitions on the Circle. (*a*) The resulting curve MBS is called a *circle*.

(*b*) It will be noticed that the opening of the compasses

does not change, hence *PM* equals *PB* equals *PS*. This length is known as the *radius* of the circle.

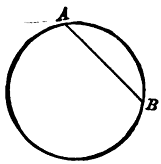
(*c*) The curve *MBSM* is called the *circumference* of the circle. The distance of the circumference from *P* is everywhere the same, *i.e.*, this distance is the radius of the circle.

(*d*) The point *P* is called the *center* of the circle.

(*e*) Any portion of the circumference is called an *arc* of the circle or of the circumference.

(*f*) Any portion of the circle cut out by two radii and the arc between them is called a *sector* of the circle. For example, in the circle above, the portion included between radii *PM* and *PB* and the arc *MB* is a sector of that circle.

(*g*) A line connecting two points on the circumference of a circle is called a *chord* of the circle. In the adjacent figure *AB* is a chord.



(*h*) A *secant* is a line cutting a circumference at two points. A chord extended both ways is a *secant.* For example, lines *AB* and *CD*, Fig. 1 below, are secants.

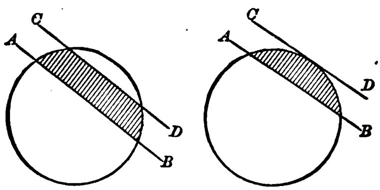


Fig. 1 Fig. 2

(*i*) A portion of a circle cut off by a pair of parallel secants is called a *segment* of a circle. One or both secants

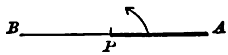
may touch, but not cut the circle, as CD in Fig. 2. The shaded portions of the adjacent circles are segments.

(j) A line that touches, but does not cut, a circle is called a *tangent,* as CD in Fig. 2.

(k) That portion of the arc of the circumference which lies between two radii forming an angle is said to be the *arc subtended* by the angle at the center, as arc MB, in the circle of §135.

137. II. *To construct an angle of 180° at a point P.*

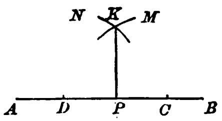
From P draw along the ruler a line to A; without lifting



ruler draw PB in direction opposite to PA. Then the angle APB is 180°, and is called a *straight angle.*

138. III. *To construct a right angle at a given point P.*

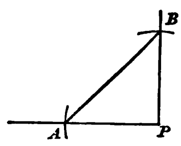
Through P draw a line AB. Place one point of the compasses at P and, with compasses unchanged, lay off two equal distances from P, PC, and PD. Place one point of the compasses at C and with radius greater than PC draw an arc M; similarly, with same radius, draw arc N from center D. From K, the intersection of the arcs, draw KP. Then the angle BPK is a *right angle.* What part of a straight angle is a right angle?



A line making a right angle with another line is said to be *perpendicular* to that line.

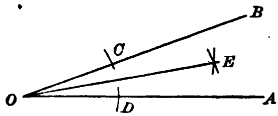
139. IV. *To construct an angle of 45 degrees.*

At P construct a right angle. Then from P make PB and PA equal by use of compasses. With ruler draw BA. Then the angle PAB will be an angle of 45 degrees.



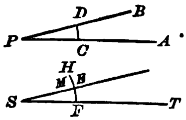
140. V. *To bisect a given angle AOB.*

Take O as the center and with as great a spread of the compasses as convenient draw the arcs C and D. Then with C and D as centers and the same spread draw arcs cutting at E. Join E and O and this line will bisect the angle AOB.



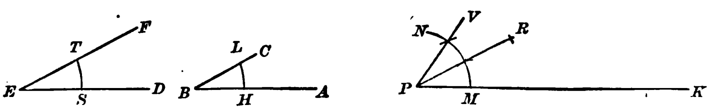
141. VI. *To construct an angle equal to a given angle.*

Let angle APB be the given angle. Draw a line ST; with center at P draw an arc meeting PA at C and PB at D. With same radius and center at S draw arc FH. Place one point of the compasses at C, the other point at D; with compasses unchanged, place one point at F and draw arc M, cutting the arc FH at E. With the ruler draw SE. Then the angle FSE is equal to angle APB and is the angle required.



142. VII. *To add two given angles, ABC and DEF.*

Draw line PK. With P as center and any convenient radius, draw arc MN. With B as center, with same radius,

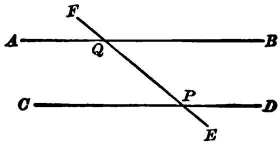


draw arc HL and with E as center and same radius draw arc ST. Now by method of § 141 construct angle KPR equal to angle ABC and angle RPV equal to angle DEF. Then angle KPV (equal angle KPR+angle RPV) is equal to the sum of ABC and DEF.

Problem: Let the student construct an angle equal to the difference between two given angles.

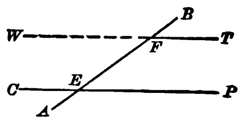
PARALLEL LINES

143. Definition. Two lines are said to be *parallel* if they make equal angles with a third line.



In the figure, AB and CD make with the line EF angle DPQ equal angle BQF; hence the lines AB and CD are parallel.

Construction. To draw two parallel lines through two points E and F.



Through E and F draw a line AB. Through E draw a line CP. At F construct on AB an angle BFT equal to angle FEP (§ 141). Produce TF to W. Then CP is parallel to WT since they make equal angles with the line AB.

PLANE SURFACES

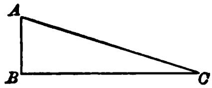
144. Definitions. The surface of a flat page of paper is an illustration of a *plane surface* or simply a *plane.*

Any plane figure which is formed by straight lines is said to be *bounded* by straight lines. Such figures are known as *polygons.*

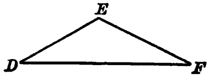
The polygons more commonly discussed are those of three and four sides. They are designated as follows:

(*a*) A polygon with three sides is called a *triangle.*

(*b*) A polygon of four sides is called a *quadrilateral.*



145. Definitions. A triangle with one right angle is called a *right triangle.* Triangle ABC is such a triangle.



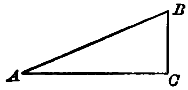
A triangle with no right angle is called an *oblique triangle.* Triangle DEF is such a triangle.

If a triangle has two of its sides equal it is called an *isosceles triangle.*

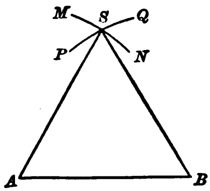
A triangle which has its three sides all equal to each other is called an *equilateral triangle* (*i.e.,* all sides equal).

146. Problems on the Triangle.

(*a*) Construct a right triangle by the method of § 138. By the method of § 142 add acute angles CAB and CBA. What result is obtained? Add the three angles of the triangle by the same method. What is the result?



(*b*) Draw an isosceles triangle with base* line AB by



taking equal radii from A and B and drawing arcs MN and PQ, intersecting at S. Connect S with A and with B. Then triangle ABS is an isosceles triangle, since it has two sides, AS and BS, equal. Now measure (using § 141) and compare the size of angles SAB and ABS. What relation is found to exist between these angles?

Add the three angles of this triangle. What is the sum found to be?

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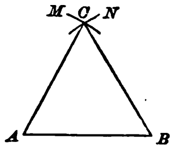
*The base line of a polygon is the line upon which the polygon seems to rest.

(*c*) Draw an equilateral triangle on AB as a base.

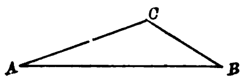
Method: With radius equal to AB and with A as center, draw arc M; with same radius and B as center draw arc N. Connect C (point of intersection of the arcs) with A and B.



Then triangle ABC is an equilateral triangle since its three sides are equal. Compare the size of the three angles of this triangle by the method of § 141. What relation exists among these three angles? Add the three angles of this triangle. What is the result?

(*d*) Draw any triangle ABC. Add the three angles and note the result. To what is the sum equal?

What was the result of adding the three angles of the triangle in each case above? What law about the sum of



angles of a triangle may be drawn from the results?

Exercise 48

1. Construct a right triangle whose legs are 5 inches and 7 inches. Measure the hypotenuse.* Check your measurement by computing its length from the formula

$$b^2+p^2=h^2$$

2. Construct a right triangle with legs 6 inches and 11 inches. Measure the hypotenuse and check as in problem 1.

3. In problem 2 draw a perpendicular from the vertex of

*The hypotenuse is the side of a right triangle opposite the right angle.

the right angle to the hypotenuse. Measure this perpendicular and both segments of the hypotenuse.

4. In problem 1 draw a perpendicular from the vertex of the right angle to the hypotenuse; measure it and the segments of the hypotenuse. In the two right triangles formed does the relation $b^2+p^2=h^2$ hold good?

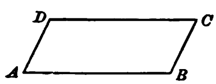
5. Draw a triangle with each of its sides 4 inches long. Draw a perpendicular from any vertex to the opposite side. Bisect all the angles. Bisect all the sides.

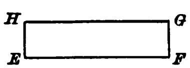
6. If your drawings are correct in problem 5 you will notice that the bisectors of the angles and the perpendiculars to the sides are the same lines and that each of them bisects the opposite side. Compute the length of these three lines. Are they all the same length?

7. Construct a triangle whose sides are 2, 5, and $6\frac{1}{2}$ inches. Bisect each angle. Draw perpendiculars from each vertex to the opposite sides (produced if necessary). Do the bisectors and perpendiculars coincide? Measure them.

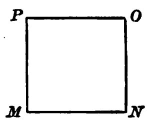
QUADRILATERALS

147. Definitions. A quadrilateral with its opposite sides equal and parallel is called a *parallelogram*. Figure $ABCD$ is a parallelogram.

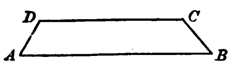




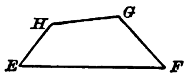
A parallelogram with its angles all right angles is called a *rectangle.* $EFGH$ is a rectangle.



A rectangle with its four sides equal is called a *square.* $MNOP$ is a square.



A quadrilateral with two of its sides parallel and two not parallel is called a *trapezoid.* $ABCD$ is a trapezoid.



A quadrilateral with no two of its sides parallel is called a *trapezium.* $EFGH$ is a trapezium.

148. Definitions on the Polygon. In a *polygon* the points of meeting of the sides are called the *vertices* of the polygon.

Angles or vertices of a polygon which are separated by only one side are said to be *adjacent* angles or vertices.

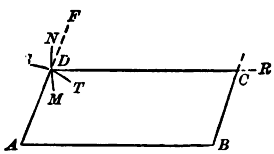
Angles or vertices of a quadrilateral separated by more than one side are called *opposite* angles or vertices. Thus, in the figure above, F and G are adjacent while F and H are opposite or *non-adjacent* vertices.

A line joining two non-adjacent vertices of a polygon is called a *diagonal* of the polygon. In the above figure, FH is a diagonal of $EFGH$.

Two polygons are said to be equal when every part of the one equals the corresponding part of the other. These corresponding parts are known as *homologous* parts of the polygons.

PROBLEMS OF CONSTRUCTION

149. *To construct a parallelogram.*



(1) Take any convenient length, AB. From B draw any suitable distance in any convenient direction, say BC. From C as a center with radius equal AB draw arc MN. From A as a center with radius equal to BC draw arc ST meeting arc MN at D. Draw CD and AD Now $AB=CD$ and $BC=AD$ because so drawn. Now show AB and DC are parallel by showing angles ADC and BCR equal; show AD and BC parallel by proving angles FDC and DAB equal. If this is

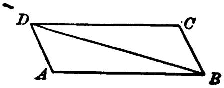
true, then, by definition, quadrilateral $ABCD$ is a parallelogram.

(2) Compare by measurement opposite angles BCD and BAD; also opposite angles ADC and ABC. What is the result in each case?

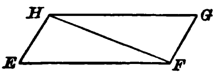
(3) Add adjacent angles BCD and ABC; also adjacent angles ABC and DAB. What is the result in each case?

(4) Perform the above operations for several parallelograms. Do the operations always give the same results? From these operations what fact can you state about the opposite angles of a parallelogram? About adjacent angles?

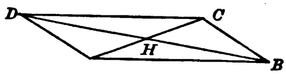
(5) In parallelogram $ABCD$ draw diagonal BD. Now compare by measurement the sides and the angles of the triangles ABD and CBD. What is the result? What relation then exists between the triangles?



(6) Draw diagonal FH in a parallelogram $EFGH$ and proceed as above. What is the result? What relation exists between the triangles in this case?



(7) In parallelogram $ABCD$ draw the two diagonals AC and BD with H the point of intersection. What relation exists between AH and HC? Between DH and HB?



Perform the operations of (7) upon several parallelograms and note whether results are always the same. Write the two general conclusions derived from these operations.

RATIO AND PROPORTION

150. Definitions. The fraction resulting from dividing a concrete number by another concrete number of the same

kind is often called a *ratio*. For example: three inches divided by five inches gives the fraction $\frac{3}{5}$, which is the ratio between 3 inches and 5 inches. $\frac{3}{5}$ is also the ratio between 3 pounds and 5 pounds, between 3 apples and 5 apples, etc. Hence it is evident that a ratio is an abstract number.

If two ratios are equal the equality thus formed is called a *proportion*. Example:

$$\frac{3 \text{ inches}}{5 \text{ inches}} = \frac{6 \text{ apples}}{10 \text{ apples}}$$

or simply $\frac{3}{5} = \frac{6}{10}$.

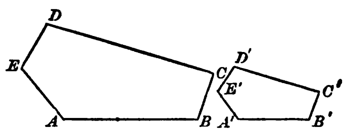
If the ratio between two quantities equals the ratio between two other quantities the four quantities are said to be *in proportion* or *proportional.*

When the ratio between two quantities equals the ratio between two other quantities and equals the ratio between still two other quantities, etc., then *all* the quantities are said to be *proportional.*

SIMILAR POLYGONS

151. Definition. When two polygons of the same number of sides have the angles of one equal to the corresponding angles of the other, and the corresponding sides proportional, then the polygons are said to be *similar*.

Polygon $ABCDE$ is similar to $A'B'C'D'E'$ if angles

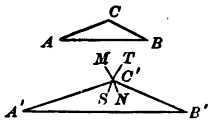


A, B, C, D, E are equal respectively to angles A', B', C', D', E', and if

$$\frac{\text{side } AB}{\text{side } A'B'} = \frac{\text{side } BC}{\text{side } B'C'} = \frac{\text{side } CD}{\text{side } C'D'}, \text{ etc.}$$

CONSTRUCTION OF SIMILAR TRIANGLES

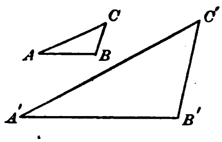
152. *To construct a triangle similar to given triangle, ABC, with sides twice as long as those of ABC.*



Method: Draw $A'B'$ twice as long as AB. Then, with A' as center and radius twice AC draw arc MN. From B' as center, with radius twice BC, draw ST intersecting MN at C'. Draw $A'C'$ and $B'C'$.

Compare the angles of A and A', B and B', C and C'. What is the result? Why are the triangles similar? (See definition of similar polygons.)

153. *To draw a pair of similar triangles.* Draw triangle ABC. With any convenient sides on $A'B'$, three times as long as AB, construct angles at A' and B' equal respectively to angles A and B and produce the sides of those angles till they meet at C'. Then compare lengths of AC and $A'C'$, of BC and $B'C'$. What relation exists between them? Are triangles ABC and $A'B'C'$ similar? Why?



AREAS OF POLYGONS

154. Definitions. By the area of a polygon is meant the number of times it will contain within its sides a unit of square measure.

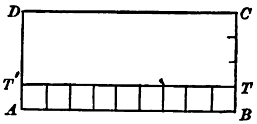
The line upon which the polygon seems to stand is called the *base* of the polygon. By considering the polygon as turned about, any side may be taken for the base.

The perpendicular distance between the base of a polygon and the highest point opposite is called the *altitude* of the polygon.

ILLUSTRATIVE PROBLEMS

155. Draw the rectangle $ABCD$ with base 9 units of length and one side 4 units long. How many square units are contained in the rectangle? *i.e.*, What is its area?

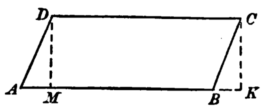
Through the first division point T draw TT' parallel to AB. From unit marks on AB erect perpendiculars to line TT'. How many square units will $ABTT'$ contain? *i.e.*, What is its area? How many such rectangles as $ABTT'$ can be made in the rectangle $ABCD$? How many square units are there, then, in rectangle $ABCD$?



Notice that side BC is the altitude of the rectangle. Perform similar operations for several rectangles of different dimensions. Note in each case the relation between the area and the length of the sides. What general law can be stated of the area of a rectangle?

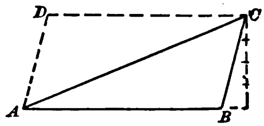
156. Find the area of a parallelogram $ABCD$, whose base is 10 units long and altitude is 4 units long.

From D draw perpendicular to AB at M and from C draw CK perpendicular to AB produced at K. What relation exists between triangles AMD and BKC? Why? What, then, is the relation between rectangle $MKCD$ and parallelogram $ABCD$? Why? How do the bases of the two figures compare? Their altitudes? What is the area of the rectangle? What, then, of the parallelogram? Find areas of several parallelograms in this way. What is the general law for the area of a parallelogram?



157. Find area of triangle ABC with base 8 units and altitude 4 units in length.

From C draw CD parallel with and equal to AB. Draw AD. What is the polygon $ABCD$? Why? What relation exists between triangle ABC and polygon $ABCD$? What is the area of the polygon? What, then, is the area of the



triangle? Why? Find areas of several triangles in this way. What do you find for the general law for the area of a triangle?

158. Conclusions. The results of the examples of §§ 155, 156, and 157 above may be conveniently summarized into the following formulas:

For rectangles or parallelograms

$A^* = bh$ (base times altitude)

For triangles

$A = \dfrac{bh}{2}$ ($\frac{1}{2}$ base times altitude)

Exercise 49

1. With ruler and compasses construct on an appropriate scale a square 8 feet on each side. Measure the length of its diagonal. Compute its length and compare.

2. Construct on an appropriate scale a rectangle with sides 200 yards and 160 yards. Measure its diagonal. Check by computing it.

3. Draw to scale a triangle with sides 8 feet, 10 feet, and 15 feet. Construct the altitude by drawing a perpendicular from a vertex to the opposite side. Compute the area of the

*A denotes the Area.

triangle. Check by constructing and measuring an altitude from another vertex.

4. Construct a square with sides 4 inches. Find its area.

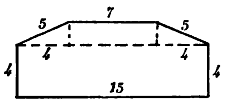
5. Construct on an appropriate scale a rectangle 2 miles long and $1\frac{1}{2}$ miles wide. How many acres in it?

6. How many square feet of surface (walls and ceiling) in a room 14 by 16 feet and 9 feet high? What will it cost to plaster it at 80 cents per square yard? (Disregard openings.)

7. In the room of problem 6 there is a window $4\frac{1}{2}$ by 6 feet and a doorway 6 by 8 feet. If these be subtracted how much will it cost to plaster at 85 cents per square yard?

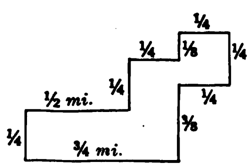
8. How much will it cost to paper the room of problem 6 at 20 cents per roll 5 yards long and 16 inches wide, subtracting nothing for doors and windows and allowing nothing for matching patterns? The border is 16 inches deep and costs 50 cents per roll of 5 yards.

9. What will be the cost of papering if the window and doorway described in problem 7 be subtracted and 20% be added for waste in matching?



10. An attic room is of the shape shown in the figure. The floor is 15 by 19 feet. Find the cost of plastering at 35 cents per square yard. The dotted lines are drawn to help solve the problem.

11. A farmer has a farm like the adjoining plot. How many acres in it? The distances indicated are all fractions of miles.



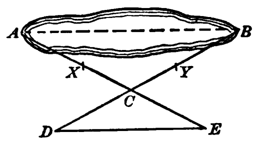
12. A triangular piece of land has sides $\frac{3}{4}$ mile, $\frac{3}{8}$ mile, and 1 mile. Plot it carefully, draw two altitudes, and compute its area two ways. *How do they check?*

13. The area of a triangle is 21.6 square feet. Its altitude is 6.4 feet. Find the base.

14. The length of a rectangle is 2 yards more than the width. The perimeter (distance around) is doubled if 3 yards be added to each dimension. Find the length and breadth. Plot both figures.

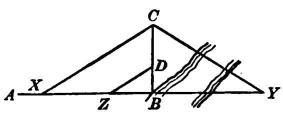
159. The principle of similar figures is very useful in computing distances.

Suppose it is desired to know the length of the lake AB. Choose a convenient point C. Drive stakes at C, X, and Y. Measure the distances CA and CB. Then using XC to give the direction, measure CE one-tenth, or any other convenient fraction of the length CA. Similarly measure CD in the line CB and equal to one-tenth CB. The join of D and E ought to be one-tenth as long as AB, since the triangles are similar. But DE can be easily measured and thus AB can be found.



It is the custom of land surveyors to find the distance across a stream or other obstacle which does not obstruct vision as follows:

Run the line AB up to the bank leaving a flagman behind at A to line up another who is sent around to Y. The surveyor then turns a right angle at B toward C and measures the angle BCY with his instrument. He then makes angle BCX the equal of BCY. It follows that $BX = BY$. The chainmen then measure BX and the survey is started again at Y. If the distance BX is very great the surveyor can drop down at D so that $BD = \frac{1}{4}BC$, then BZ will be $\frac{1}{4}BY$, thus saving time in measuring BX.

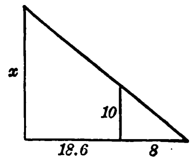


Exercise 50

1. A triangle has sides 6, 7, and 4. A line is drawn parallel to the side 4, dividing the other sides in halves. Find the length of this line.

2. In problem 1, suppose the line is drawn so as to cut the side 6 into two parts 4 and 2 and the side 7 into two parts $\frac{14}{3}$ and $\frac{7}{3}$. How long is this line?

3. Wishing to determine the height of a barn a boy sets up a stake as shown in the figure and measures the distances as indicated. What is the height, x, of the barn? The numbers represent feet.



4. To find the height of a tall tree show how you can use the shadow of the tree and the shadow of a 10 ft. pole.

5. How high is a tree which casts a shadow 135.6 feet long when a 10 foot pole casts a shadow 11.3 feet long?

6. An equilateral triangle of side 10 feet is to be quadrupled in area but unchanged in shape. What will be the length of the side of the new triangle?

7. Drop a perpendicular from the vertex of a right triangle on the hypotenuse. Measure all the lines carefully. Are the two triangles thus formed similar? How do their areas compare when the hypotenuse is 13 feet long and the legs 5 and 12 feet respectively?

CHAPTER VIII

POLYNOMIALS

DEFINITIONS

160. Method of Expressing Numbers. A number may be expressed by figures, by letters, or by a combination of letters and figures. Such a number is called an *algebraic number*, an *algebraic expression*, or simply a *number*.

An algebraic number with none of its parts separated by plus or minus, together with the sign preceding it, is called a *term*.

For example, a^2, $-5ab$, $\frac{1}{2}x^2y$, and $-3xyz$, are terms.

161. Classification by Number of Terms. An expression consisting of a single term is called a *monomial*.

If an expression is made up of two terms separated by a plus or a minus sign, it is called a *binomial*.

Thus, $3a+4b$, $4a^2-2c$, $4+11b$ are binomials.

A number composed of three terms separated each from the other by plus or minus signs is called a *trinomial*.

Thus $3x+2-a$, $4ab+2abc-4$, and $m-2n+c$ are trinomials.

Algebraic expressions are more commonly classified, however, as numbers of *one* term and numbers of *more* than one term. In this classification, all expressions of more than one term are called *polynomials* (*many terms*). Thus, binomials and trinomials are polynomials of two and three terms, respectively.

A polynomial consists of two or more monomial terms.

If a number expression is put into a parenthesis it is con-

sidered as one term. For example, $3x+(y+z)$, though a binomial, is equivalent to the trinomial $3x+y+z$.

162. Similar Terms. Two or more terms are similar with regard to a certain letter if that letter occurs with the same exponent in each term.

For example, $2a, -7a$ are similar in a. So also $2ax, -7ay^2$ are similar in a.

The last two expressions are not similar as to x or y. Similar terms may be combined by adding their coefficients according to § 50.

Two or more terms may be similar in more than one letter, as $-\frac{3}{2}ax$, $7axy$ are similar in a, in x, and in ax.

Two or more terms may be similar in having a common factor, as $3w(a-x)$, $-\frac{3}{7}k^2(a-x)$ are similar in $(a-x)$.

Exercise 51

Combine the following:

1. $2a-3a+7a$.

$2a-3a+7a=2a+7a-3a=9a-3a=6a$.
Or, using the parenthesis as explained in § 89,
$2a-3a+7a=a(2-3+7)=a\cdot 6=6a$

2. $3x-2x+5x$
3. $7a-3a-2a$
4. $3a-2a+4x-3x$
5. $-4x+5x-7y+3y$
6. $9t-6t+4t-3t$
7. $-5w-6w-w$
8. $+2a-5a-2x+5x$
9. $3a^2-7a^2+4a^2$
10. $ax+bx+cx$
11. $2xy-3x+4xw$
12. $b+br$
13. $F+Frt$

ADDITION OF POLYNOMIALS

163. Laws of Operation. It is agreed that the laws of *operation in* arithmetic and algebra shall be the same. Then

in algebra it is allowable to arrange terms that are to be added or subtracted in any order since this is possible in arithmetic.

For example, $3+2$ is the same as $2+3$. Hence we say $3x+2x$ is the same as $2x+3x$. In general, $3-2+7-5=7-2-5+3$. Hence $3xy^2-2z+7a-5c=7a-2z-5c+3xy^2$.

This principle is called the *commutative law* of algebra. It is stated formally below for reference.

164. Commutative Law. *If two or more terms are to be added or subtracted, they may be arranged in any order, provided the sign preceding each term is always carried with it.*

165. Associative Law. In arithmetic the order in which additions and subtractions are performed is immaterial.

$$e.g.,\ 2+8-3=(2+8)-3=2+(8-3)$$

that is, we can add the 2 and 8 and subtract 3 from the sum or we can subtract 3 from 8 and add 2 to the difference. The result is the same. (Let the student make other examples of this nature.) This principle is true also in algebra and is stated as follows:

If two or more terms are to be added or subtracted they may be combined in any order.

This is called the *associative law* of addition.

Note that these laws apply as yet only to addition and subtraction.

Since the terms of a polynomial may be arranged in any order we have the following rule:

166. Rule. *To add two or more polynomials write them in columns with like terms under like terms and combine these like terms according to* § 50.

For example, to add $2a-b+3c$, $4a+2b-7c-5y$, $-3a+2b+10c+6x$.

$$\begin{array}{rrrrr}
2a-&b+&3c&&\\
4a+&2b-&7c&-5y&\\
-3a+&2b+&10c&&+6x\\
\hline
3a+&3b+&6c&-5y&+6x
\end{array}$$

167. Method of Checking. Since the letters in any algebraic expression represent any numbers whatever, the addition above ought to be true for any values of a, b, c, x, and y. The simplest are $a=1$, $b=1$, $c=1$, $x=1$, $y=1$. When these values are substituted in the addends and combined, the result ought to come out the same number as when substituted in the sum of the three parts.

An example will illustrate the solution and its check:

Add $a^2-4a+10$, $5a-6a^2+4$, $3a-16+2a^2$.

Solution:	Check:	
$a^2-4a+10$	if $a=1$, $1-4+10=$	7
$-6a^2+5a+4$	if $a=1$, $-6+5+4=$	3
$2a^2+3a-16$	if $a=1$, $2+3-16=$	-11
$-3a^2+4a-2$		-1

But $-3a^2+4a-2$, if $a=1$, is also -1.

168. A numerical check of this sort is valuable in teaching the student to rely on himself for his accuracy. Care must be taken that the check is not "forced." The two parts of the check must be performed independently, and if the results do not agree you may be sure there is something wrong. It should be pointed out, also, that it is possible that errors occur which such a check will not detect.

Thus, if the sum had by mistake been written $-3a^2+3a-1$, the results would have apparently checked. In actual practice it is very seldom that an incorrect result checks.

All work of this nature should be checked by numerical substitution during the first year, not to please the teacher, but to get the habit of self-reliance.

Exercise 52

Add the following and check:

1. $a+b-c$, $2a-b+3c$, $7a+3b-4c$
2. $5a-b+2c$, $-8a+6b-3c$, $2a-b+3c$

3. $5x+y,\ 6x-7y+4z,\ 5z+3x-3y$

4. $7m+n+2p,\ -5m+6n,\ 3m+5n-6p$

5. $5x-4y+3z,\ 2y-3x+z,\ 3z-2y-4x$

6. $14x-3z+y,\ 13y-z+10x,\ 6z-9y$

7. $8m-7n-p,\ 5m,\ -3n+4p,\ 3n-18p$

8. $8ax-2xy,\ 2ax+6ay,\ +11ax+9ay$

9. $3x^2-14a-10,\ 3a-4+6a^2,\ -6+12a^2-a$

10. $\frac{1}{3}a+\frac{2}{3}b+\frac{1}{2}c,\ 2a-3b-7c,\ \frac{2}{3}a+\frac{4}{3}b-\frac{3}{2}c$

11. $\frac{2}{5}a+5b+2c,\ \frac{1}{10}a-\frac{1}{7}b,\ b-\frac{9}{11}c.$

12. $12x^2+6b^2+4c^2,\ 9b^2-2a^2+17c^2,\ 3b^2-2c^2$

13. $20xy-x^2,\ -6y^2+13xy-7x^2,\ -15y^2-30xy+12y^2$

$ax+bx+cx$ can be written $(a+b+c)x$. Write the following with polynomial coefficients:

14. $2xy+4y-cy$

15. $16ax-4xy+2x$

16. $13xy^2+cy^2-10y^2+x^2y^2$

17. $bx+4cx-6x+4bx$

SUBTRACTION OF POLYNOMIALS

169. Method for Subtraction. In §§ 51, 52 we studied the subtraction of signed numbers and arrived at the rule:

"*To subtract one quantity from another change the sign of the subtrahend and proceed as in addition.*"

An extension of this is applied to the subtraction of one polynomial from another.

170. Rule. *To subtract one polynomial from another place like terms in the same column, change all the signs in the subtrahend, and proceed as in addition.*

171. Checking. Subtraction should be checked by assigning convenient values for the letters involved, as in addition.

For example, from $13xy+2a-7rs$ subtract $-5xy+3ax-7a$.

Solution:

$$\begin{array}{l} 13xy+2a-7rs, \\ -5xy-7a \quad\quad +3ax, \\ \hline 18xy+9a-7rs+3ax \end{array}$$

Check:

$$\begin{array}{lr} \text{if } x=1, y=1, a=1, r=1, s=1, & 13+2-7= \quad 8 \\ \text{if } x=1, y=1, a=1, & -5-7+3=-9 \\ \hline & +17 \end{array}$$

For the same values of the letters the difference $=18+9-7+3=+23$.

Therefore there is some error in the work. Looking over it we see that we failed to change the sign before $+3ax$ when subtracting. If this change be made the results will be found to check.

Exercise 53

1. From $8a+4b-6c$ subtract $3a-7b+2c$
2. From $10xy-3rs+11ab$ subtract $6xy-12rs+13ab$
3. From $-9ax-14ad-2ay+15z$ subtract $-16z+2ad+5cy$
4. From $3t+2r-5mn-2ab$ subtract $-2t-4r+8m$
5. From $31b-4xy+16ax-1$ subtract $8a-6cd+3v$
6. From $3\frac{1}{2}x-2$ subtract $12\frac{1}{3}x-\frac{3}{4}$
7. From $3.55x-2.1a$ subtract $6.4y-1.7x$
8. From $7xca-4by+6cy$ subtract $10-3acx-6yb+5yc$
9. From $abc+2(a+b)-3xa+4$ subtract $6-4ax+3(a+b)$
10. From $6.10-3x$ subtract $3.10-x$
11. If $P_1=\$300r$ and $P_2=\$100r$ find P_1-P_2
12. If $I_1=\$600\ (.08t)$ and $I_2=\$200\ (.08t)$ find I_1-I_2
13. If $A_1=\$300+\$300r$ and $A_2=\$100+\$100r$, find A_1-A_2
14. If $P_1=F+\frac{1}{4}Fr$ and $P_2=F+\frac{1}{6}Fr$ find P_1-P_2

15. If $P_1=F+\frac{1}{3}Fr$ and $P_2=2F+\frac{2}{3}Fr$ find P_1-P_2

16. From $2(b+c)-4(m+n)$ subtract $-6(a+b)+5(m+n)$

17. From $31(b+c)+16(m+n)$ subtract $9(m+n)-16(a+b)$

18. From $a(b+c)+d(m+n)$ subtract $6(a+b)+2(m+n)$

19. From $2a(b+c)+3d(m+n)$ subtract $a(b+c)-d(m+n)$

20. From $x(b+c)+y(m+n)$ subtract $a(b+c)-d(m+n)$

SIGNS OF AGGREGATION

172. Definitions. In Chapters V and VI, and in the present chapter we have made some use of a parenthesis. Now we give definitions of several symbols of the same nature as a parenthesis and rules for their use.

The *parenthesis* ( ), the *brace* { }, and the *bracket* [ ] are called *symbols of aggregation* because the whole expression inclosed in them must be regarded as one number.

Any operation applied to a number in a parenthesis, a brace, or a bracket must be applied to the whole quantity, not to a part of it.

Thus, $2(a+b)$ means $2a+2b$, while $2a+b$ means that a alone is to be multiplied by 2.

The parenthesis, the brace, and the bracket all mean the same thing, and are frequently spoken of as parentheses.

For example, $2(a+b)$, $2\{a+b\}$, $2[a+b]$ are identical.

Why we need so many symbols for the same operation will appear later.

173. Removal of Parentheses. In the solution of equations and in other operations of algebra it is frequently necessary to remove all parentheses. A parenthesis preceded by the sign + means that each term in it is to be added to the terms preceding the parenthesis. A parenthesis with a $-$sign before it means that each term within it is to be sub-

tracted from the terms preceding the parenthesis. Hence we have the rules:

174. Rules. *A parenthesis with a + sign in front may be removed without any change in the signs of the terms inside.*

A parenthesis with a − sign before it may be removed by changing the sign of each term within it.

A parenthesis preceded by a number may be removed by multiplying every term within by that number. The method of performing this multiplication is given in the following sections. If no sign is before a parenthesis, then the + sign is understood. If one parenthesis is inside another, the beginner should remove the inside one first and simplify the result before removing the others.

175. Insertion of Parentheses. A knowledge of the insertion of parentheses is just as necessary and as helpful as a knowledge of their removal. The rules for removing parentheses give us the following obvious rules:

176. Rule. *To insert a parenthesis with a + sign before it no change is necessary.*

To insert a parenthesis with a − sign before it, every term put within must have its sign changed.

For example, remove the parentheses from $2x-(5y-2z)$.

$2x-(5y-2z)=2x-5y+2z$, since every sign in the parenthesis must be changed.

Insert a parenthesis about the last two terms of $2x-5y+2z$.

$$2x-5y+2z=2x-(5y-2z)$$

From these examples, it is clear that two successive changes of sign leave the expression the same as it was at first.

For example, remove the parenthesis from:

$$(5x-6y)-\{-2x-(4y+z)+3z\}.$$

Solution:

$(5x-6y)-\{-2x-(4y+z)+3z\}=5x-6y-\{-2x-4y-z+3z\}=$
$5x-6y-\{-2x-4y+2z\}=5x-6y+2x+4y-2z=7x-2y-2z.$

Exercise 54

Remove parentheses and combine similar terms:

1. $10-(3-2)+5$
2. $10x-(3x-2x)+5x$
3. $10a-(3x-2y)+5z$
4. $(2a-3b)-(4a-5b-6)$
5. $(a+b)-(4b-2a)-(a+3b)$
6. $10t-(4+3t)+(16t-6)$
7. $2a-(6a-3b+c)-(4c-7b+8)-(a+5)$
8. $5r-(7-3r+6s)-(4r+2)+(4s-2r-62)$
9. $\{2x-(y-z)\}$
10. $\{-(4y-x)-3x\}$
11. $\{2x-(y-z)\}-\{-(4y-x)-3x\}$
12. $14ab-(3b-31a+71ab)-(-4a-69ab+28a)$
13. $14ab-\{(3b-31a+71ab)-(-4a-69ab+28a)\}$
14. $14ab+\{(3b-31a+71ab)-(-4a-69ab+28a)\}$
15. $a-3(a+b)=a-3a-3b=-2a-3b$
16. $2x-3(x+6)+(3a-2b)$
17. $6a-14(a-b+2c)+2\{3(2x-1)\}$
18. $\{-2(3a-b-c)+(2a-6b+c)\}+4a$
19. $\{-a(3a-b-c)+3a(2a-6b+c)\}+4a(b+c)$

In each of the following insert a parenthesis around the last three terms (1) with + sign before it, (2) with − sign before it. Check your work by removing the parentheses.

20. $4x+3y-2c+2d$
21. $9xy+yz-5ax-3$
22. $6r-3s+c-2t$
23. $11-3x-c+21g$
24. $12ab-19bx+15ax-4r$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

177. Signed Numbers. The student should review §§ 61 and 62 on the multiplication of signed numbers. The principles explained there will be in constant use here and hereafter.

178. Laws of Operation. We assume that the following three laws govern all multiplications:

Law 1. *The factors of a product may be written in any order.*

Thus, $a \cdot b = ba$, and $x(a+b) = (a+b)x$.

This is called the *commutative law* of multiplication. This is in the spirit of our former agreement, § 163, that all the rules of arithmetic shall apply in algebra, for in arithmetic it is evident that $7 \cdot 6 = 6 \cdot 7$, etc.

Law 2. *The multiplications in any product may be performed in any order.*

Thus, $abc = a(bc) = (ab)c$, or

$$2a^2(x-y) = 2\left\{a^2(x-y)\right\} = (2a^2)(x-y)$$

This, again, is only a precise statement of what we do every day in arithmetic. For in the product $6 \cdot 2 \cdot 3$ we may multiply the 6 by the 3 and this product by 2 as $6 \cdot 3 \cdot 2 = 18 \cdot 2 = 36$, or we may say $6 \cdot 3 \cdot 2 = 6 \cdot 6 = 36$.

This is called the *associative law* of multiplication.

Law 3. *To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and write the results in succession with their proper signs.*

For example, $x(a+b) = ax+ab$; $7a(2a-3b) = 14a^2-21ab$.

This is also a law of arithmetic, but in arithmetic it is not so frequently used as are the commutative and associative laws. For example, 2 (3+4) may be found in two ways as follows:

$$2(3+4) = 2(7) = 14, \text{ or}$$
$$2(3+4) = 2.3+2.4 = 6+8 = 14$$

The second way is not common in arithmetic, but it is just as correct as the other and it is the more common method of procedure in algebra.

This is called the *distributive law of algebra.* It must be noted that in arithmetic these three laws were used only for positive numbers, but in algebra we assume their validity for all kinds of numbers, integral, fractional, positive, and negative.

179. Illustration. The work of multiplying a polynomial by a monomial is arranged as follows:

$$\begin{array}{ll} \text{Multiply} & 2a^2-ab+3b-6a-7 \\ \text{by} & 3ab \\ \hline & 6a^3b-3a^2b^2+9ab^2-18a^2b-21ab \end{array}$$

Check: This multiplication must hold true for any values of a and b. Hence, let $a=2$ and $b=3$. Then

$$\begin{array}{lr} 2(2)^2-(2)(3)+3(3)-6(2)-7=2.4-6+9-12-7= & -8 \\ 3(2)(3) \quad = & +18 \\ \hline & -144 \end{array}$$

$$\begin{array}{l} \text{But } 6(2^3)(3)-3(2^2)(3^2)+9(2)(3^2)-18(2^2)3-21(2)(3)= \\ 6(8)(3)\ -3(4)(9)\ +9(2)(9)\ -18(4)3\ -21(2)(3)= \\ 144\ \ -108\ \ +162\ \ -216\ \ -126\ \ =-144, \end{array}$$

which checks the work.

If we put $a=1$, $b=1$, we get also a fairly good check more easily, but it must be noted that $a=1$, $b=1$ will not check the accuracy of the exponents, since a^3 is the same as a^2 and a if 1 be substituted for a. This is not true, however, if 2 be substituted for a. $a=2$, $b=2$ would have given an easier check than $a=2$, $b=3$.

Exercise 55

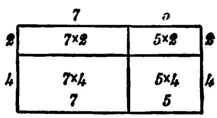
Multiply and check:

1. $a+2$ by $3a$
2. a^2+6 by a
3. $3a^2+6a$ by 7
4. $2x^2-17x$ by $4x$

5. $5x^2+2x-1$ by $-x$
6. $5x^2+2x-1$ by $-3x^2$
7. $a^2-3ab-b^2$ by $2ab$
8. $-6y^2+17ay^3+31a$ by $6xy^2$
9. $a^4-a^2b^2-6b^4$ by $-4ab^3$
10. $6a^3-21a^2+17a-41$ by $13a^2b$
11. $4r^2-\frac{2}{3}$ by $3r$
12. r^2-r+1 by $\frac{1}{3}rs$
13. $\frac{1}{2}s^3-\frac{4}{5}-s-\frac{13}{9}$ by $20t^3$
14. $.5x-3.1$ by $2x$
15. $-.3a^2+6.3a-.04$ by $-.6a^2$
16. $b+.06b$ by $3t$
17. $F+.08Ft$ by $.02$
18. $b(1+r)=?$
19. $-3a(ab-6bx+1.7cx^2)=?$
20. $-3.4rt(21b^2+.07b-34)=?$

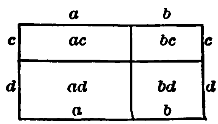
THE PRODUCT OF TWO POLYNOMIALS

180. Illustrative Examples. Suppose a rectangular garden plot 12 feet by 6 feet is cut into four rectangular parts as shown in the figure. Its area can be found as a whole by multiplying the length by the breadth.



Thus, $(7+5)(2+4)=$ area in square feet. The area can be found piece at a time by adding the four small rectangles $7\times2+7\times4+5\times2+5\times4$. The two areas are certainly equal, therefore $(7+5)(2+4)=7\times2+7\times4+5\times2+5\times4$.

In the language of algebra we may say that a rectangle,



whose sides are a, b, c, and d, is divided into four parts, as shown. Then, as in the numerical example, we have

$$(a+b)(c+d)=ac+ad+bc+bd.$$

This is the general formula for multiplying two binomials. It can easily be extended to a binomial times a trinomial, the product of two trinomials, etc., as indicated in the adjoining figures.

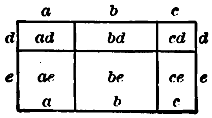


Fig. 1

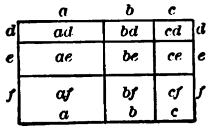


Fig. 2

$(a+b+c)(d+e)=ad+ae+bd+be+cd+ce$ (See Fig. 1).

Fig. 2 illustrates the following equation:

$(a+b+c)(d+e+f)=ad+ae+af+bd+be+bf+cd+ce+cf$

The general principle covering all these cases is the following rule:

181. Rule. *To multiply one polynomial by another, multiply the first by each monomial term of the other and add the results.*

182. Negative Numbers. In the illustrative examples we used only positive numbers, but the rule is assumed to hold for all numbers both positive and negative. If, after the partial products have been formed, any like terms appear they should be combined into one as in § 50.

For example, to multiply $3a-7x+b$ by $2b-a$.

Solution:
$$\begin{array}{l}
3a-7x+b \\
\quad\;\; 2b-a \\
\hline
6ab-14bx+2b^2 \\
-ab \qquad\qquad\quad -3a^2+7ax \\
\hline
5ab-14bx+2b^2-3a^2+7ax
\end{array}$$

Check: $a=2,\ b=2,\ x=1$

$$\begin{array}{ll} 3(2)-7+2 & =1 \\ 4-2 & =\underline{2} \\ & 2 \end{array}$$

$$\begin{array}{llllll} 6(2)(2) & -14(2) & +2(2^2) & -3(2^2) & +7(2) & = \\ 24 & -28 & +8 & -12 & +14 & =6 \end{array}$$

The two results are not the same, therefore there is some error. Looking over the work we find that in the check we have accidentally put 6(2)(2) instead of 5(2)(2). When the correction is made the two results agree.

Exercise 56

Multiply and check:

1. $(2a+b)(x+2y)$
2. $(x-2y+3)(6x-y)$
3. $(20x+16a-21b)(16+2x+a)$
4. $(a-x)(8x+3a)$
5. $(a-3x)(9x+3a)$
6. $(6x+4y-5x-2y)(6y+4x-3y+2y)$
7. $(ab-5xy+4a)(6xy-4-7b)$
8. $(x-13t)(x-t+7)$
9. $(10b-a-8b)(3a-3b-2a)$
10. $(\frac{3}{2}x+y)(x+y)$
11. $(\frac{1}{2}x+a)(x+\frac{1}{2}a)$
12. $(\frac{7}{8}x-\frac{1}{2}y+a)(\frac{2}{3}x-\frac{1}{4}y)$
13. $(\frac{3}{4}a^2-\frac{1}{2}a-1)(\frac{4}{5}a-\frac{2}{3})$
14. $(\frac{1}{2}x+y)(\frac{3}{4}-y)$
15. $(.5x+y)(.75-y)$
16. $(10.2a^2-1.4b+.06)(2.1a^2-1)$

DIVISION OF A POLYNOMIAL BY A MONOMIAL

183. The division of one monomial by another was *discussed in* Chapter III. The student should review the

last article of that chapter and solve again some of the problems. Particularly the law of exponents must be well known. Division is always the reverse of multiplication. Since $x(a+b+c)=ax+bx+cx$, it follows that $(ax+bx+cx)\div x=a+b+c$. Hence the rule:

184. Rule. *To divide a polynomial by a monomial divide each term of the polynomial by the monomial.*

185. Indicated Division. In case there are no factors in any terms which will cancel, the division cannot be performed but only indicated, thus,

$$(ab+cy+dc)\div xt^2=\frac{ab+cy+dc}{xt^2}$$

Checking. Let the pupil devise a method for checking such an exercise as the one just given.

186. Summary of Chapter VIII. In the first seven chapters of this book the student learned to use the simpler algebraic numbers. The present chapter takes him a step further into the science of algebra, as follows:

The names of the various kinds of algebraic numbers.

The operations of addition, subtraction, multiplication, and division of these numbers.

The signs of aggregation are defined and studied, §§ 172, 174, and 176.

All these operations and symbols are needed in the next two sets of exercises. The final test of a student's knowledge lies in his ability to apply it to problems.

Exercise 57

Perform the following indicated divisions:

1. $\dfrac{4a^2-8ab}{2a}$ **2.** $\dfrac{6a^3-18a}{-6a}$

3. $\dfrac{16x^2-12x^3}{4x}$

4. $\dfrac{-3xy+27y^3}{-3y}$

5. $\dfrac{9r^3t-21r^4}{3r^2}$

6. $\dfrac{49ab^3-35b^2}{7ab^2}$

7. $\dfrac{3c^2d-24c^4d^2+18c^3d}{4c^2d}$

8. $\dfrac{xyz-y^2z^2}{x^2yz}$

9. $\dfrac{-x^4y-xy^4-ax^2}{x^3}$

10. $\dfrac{65abc-52a^2b^2c^2+39a^3b^3c^3-91xab}{13a^2bc}$

11. $\dfrac{4(x-4)+2(x-4)}{x-4}$

12. $\dfrac{6(2x-1)-3a(2x-1)}{3(2x-1)}$

13. $\dfrac{6x^4-9x^{2a}}{3x^2}$

14. $\dfrac{1.21x^3+.44ax^2}{1.1x^2}$

15. $\dfrac{6x^{4a+1}-9x^{2a+1}}{3x}$

16. $\dfrac{\frac{1}{3}a^2-\frac{2}{5}a^3y-\frac{7}{21}ay^2}{4ay}$

17. $\dfrac{-24r^6-4sr^3+4.8a^3r^{10}}{.8r^4}$

18. $\dfrac{x^a+x^{a+1}-x^{a+2}}{x^2}$

19. $\dfrac{5x^{a-3}-10x^{2a-7}-20x^{4a+3}}{.5x^{2a-3}}$

20. $\dfrac{2.5x^{2n+7}-35x^{6n+1}-2.05x^{n-8}}{-.05x^{-2n+1}}$

EQUATIONS INVOLVING THE PARENTHESIS

187. Signs of Aggregation in Equations. The signs of aggregation occur so frequently that it is well for the student to learn how to handle them in a few equations. The rule of *operation* is to remove all parentheses and then solve as in

Chapter IV. Sometimes the square of the unknown quantity will appear and then be cancelled as in the following example:

188. Illustration.

Solve: $1+(x-1)(x-3)=7-(2-x)(1+x)$

$$1+x^2-4x+3=7-2-x+x^2$$

Collecting, $x^2-4x+4=x^2-x+5$

Subtracting, $\underline{x^2-x+4=x^2-x+4}$

$$-3x = 1$$

Therefore, $x = -\frac{1}{3}$

Check: $1+(-\frac{1}{3}-1)(-\frac{1}{3}-3)=7-(2+\frac{1}{3})(1-\frac{1}{3})$

$$1+\frac{4}{3}\cdot\frac{10}{3}=7-\frac{7}{3}\cdot\frac{2}{3}$$

$$1+\frac{40}{9}=7-\frac{14}{9}$$

$$\frac{49}{9}=\frac{49}{9}$$

Exercise 58

Solve and check:

1. $3(x-2)=9$
2. $3(x-1)+1=7$
3. $6(2-x)+(x-1)=16$
4. $6r-3(3r-1)-1=0$
5. $3s-6(2s+3)=2(s-1)$
6. $6y-11-3(y-4)=y-21$
7. $5t-10(n+3)=3(2n-3)$
8. $3(x-8)-2(5x-7)+3=3x-2$
9. $2h-2(6h-17)=3(2h-6)$
10. $(r-6)(r-1)=(r+6)(r+1)$
11. $(x+13)(3x-2)=3(x-1)(x+3)$
12. $2(x+4)(4x+7)-8x^2-63=0$
13. $6(2y-1)+6y-4(y-7)+14=0$
14. $10(.2y-1)+6y-40\left\{\frac{y}{10}-.7\right\}+1.4=0$

Exercise 59—Problems Involving Parentheses

1. A line 10 feet long is divided into two parts. If one part is 6 the other is $10-6$ or 4. If one part is x what part is the other? If one part is $2x$ what is the other?

2. If each of three horses cost $100 what is the cost of all?

3. If each of n horses cost $100 what represents the cost of all?

4. A horse and a cow together cost $200. The cow costs $40. What does the horse cost?

5. A horse and a cow together cost $200. The cow costs d dollars. What does the horse cost?

In some of the above exercises the answer is a binomial. For example the answers to the second and third parts of problem 1 are $(10-x)$ and $(10-2x)$. In solving a problem these quantities would be put in parentheses because they must be considered one quantity. In the following problems there are two unknown quantities. One of the unknowns can be represented by a single letter and the other can be represented as a binomial involving this letter. Much practice will be needed to learn to think of and handle a binomial as though it were a single letter.

6. The sum of two numbers is 21. Four times the smaller equals twice the larger plus 6. What are the numbers?

Let s = the smaller number.

Then, since the sum of the two is 21 the other number must be $(21-s)$

$4s$ = four times the smaller

$2(21-s)$ = twice the larger

$2(21-s)+6$ = twice the larger plus six.

$$
\begin{array}{lrl}
\text{Therefore,} & 4s = & 2(21-s)+6 \\
 & 4s = & 42-2s+6 \\
\text{Collecting,} & 4s = & -2s+48 \\
\text{Subtracting,} & -2s = & -2s \\
\hline
 & 6s = & 48 \\
 & s = & 8 \\
 & 21-s = & 13
\end{array}
$$

Check: Four times the smaller = 32.

Twice the larger plus 6 = 26 + 6 = 32.

7. The sum of two numbers is 31. Three times the smaller one equals the larger plus 1. Find the numbers.

8. Separate 49 into two parts such that 9 times the smaller part equals twice the larger part plus 1.

9. The sum of two numbers is 78. Three times the greater equals five times the less plus 90. Find the numbers.

10. The sum of two numbers is 14. Nine times one minus eleven times the other equals zero. Find the numbers.

11. The product of two consecutive numbers equals the product of the next two consecutive numbers minus 22. Find all the numbers.

12. The product of the largest and smallest of four consecutive numbers equals the product of the other two minus two. Find the numbers.

13. The value of 21 pieces of money consisting of nickels and dimes is $1.50. Find the number of each.

14. The value of 50 coins consisting of pennies and dimes is $2.39. Find the number of each.

15. A sum of $1200 is invested partly at 6% and partly at 5%. The income is $70 per year. Find the amount invested at each rate.

16. A part of $1700 is invested at 8% and the remainder at 7%. The total income is $125. Find the amount invested at each rate.

17. Two sums of money, one at 5% and one at 7%, bring an annual income of $36.75. One sum exceeds the other by $75. Find each.

Exercise 60—Physical Problems

In textbooks on physics the distance passed over by a moving body is called *space* and is denoted by the letter s.

The rate of motion, that is the distance moved per unit of time, is called *velocity*. It is represented by v. The number of units of time is represented by t. If an athlete runs 30 feet per second, how far will he run in 10 seconds? 20 seconds? Sound travels through the air at the rate of 1080 feet per second. How far will it travel in 4 seconds? In 7 seconds? The space is found by multiplying the velocity by the time. Hence the formula

$$\text{(I)} \quad s = vt$$

The formula is true if the velocity is the same one second as it is the next.

1. Solve Equation I for v in terms of s and t.

2. A boy watching a pile driver observes that it takes the sound 5 seconds to reach him after he sees the driver hit the pile. How far away is the pile driver? What formula did you use?

3. Watching an engine approach a crossing I saw the steam from the whistle 16 seconds before the sound reached me. How far away was the engine?

4. The flash from a stroke of lightning was seen 37 seconds before the thunder was heard. What was the distance?

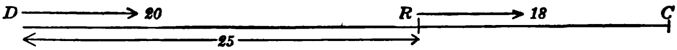
5. A cannon ball traveled 10 miles in $17\frac{1}{2}$ seconds. What was its average velocity?

6. An athlete ran a quarter of a mile in 49 seconds. What was his average velocity? He was observed to take 147 steps. How far did he step each time?

7. It takes a ray of light 500 seconds to travel from the sun to the earth (93,000,000 miles). What is the velocity of light?

8. A dog runs 20 yards per second and a rabbit 18. If the rabbit has 25 yards start how long will it take the dog to catch up?

Let the positions of the dog and the rabbit at the start and finish be as indicated in the figure.



Let DC $=s_2$ and RC $=s_1$. Then by Formula I, $s_1=18t$ and $s_2=20t$. From the figure $s_2=s_1+25$

Therefore, $20t=18t+25$

Subtracting, $18t=18t$

$2t= \quad 25$

$t=12\frac{1}{2}$

The dog will catch up in $12\frac{1}{2}$ seconds.

9. In problem 8 suppose the rabbit runs with velocity v_1 and is a feet in advance of the dog which has velocity v_2. Draw a figure and verify that the following equation holds true:

$$\text{(II)} \quad v_2t=v_1t+a$$

10. Solve Equation II for a in terms of v_1, v_2, t.

11. Solve Equation II for t in terms of a, v_1, v_2.

12. In a quarter mile race a fast runner is set back 9 feet for a false start. If he can run 24 feet per second how long will it take him to overtake an opponent who runs only 23 feet per second?

13. A dirigible going 60 miles per hour, is pursued by an airplane which travels 105 miles per hour. If they are together in 20 minutes how much of a start did the dirigible have? (Use your solution of Formula II.)

14. Solve problem 13 by substituting directly in Formula II and solving the equation for a.

15. A freight train leaves Memphis for Chattanooga at 6 A.M. and averages 20 miles an hour. A passenger train waits till the freight is 50 miles away and then follows at 32 miles per hour. At what time will it overtake the freight? Use the result of problem 11.

16. The velocity of the minute hand of a clock is 1 (revolution per hour). The hour hand goes $\frac{1}{12}$ as fast. How many minutes after 2 o'clock will the two hands be together? Use the result of problem 11.

17. When between 6 and 7 P.M. will the two hands be together?

18. When between 3 and 4 A.M. will the two hands be together?

19. When between 9 and 10 o'clock will the minute hand be 20 minute spaces behind the hour hand?

CHAPTER IX

MULTIPLICATION BY THEOREM

189. Definition. A fundamental truth in algebra is called a *theorem*. Such a truth expressed in algebraic language is called a *formula*.

190. Use of Theorems. By the use of certain theorems the labor of performing multiplications may often be greatly reduced.

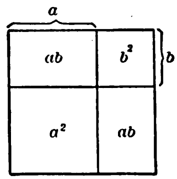
Below are given a number of very important theorems which later are applied and used in various ways.

191. Theorem I. *The square of the sum of two quantities is equal to the square of the first plus twice the product of the first and the second plus the square of the second.*

For example, let a and b be the two quantities.

Then $(a+b)(a+b)$ gives

$$\begin{array}{rl} & a+b \\ & a+b \\ \hline & a^2+ab \\ & \quad\;\; +ab+b^2 \\ \hline & a^2+2ab+b^2 \end{array}$$



This may be shown graphically as follows: Let a and b denote the lengths shown in the figure. Then $(a+b)^2$ is as shown in the adjacent figure. The square a^2, the two rectangles ab and ab, and the square b^2 making the $(a+b)^2=a^2+2ab+b^2$.

Exercise 61

Using the above theorem write the squares of the following:

1. $(2a+b)^2$
2. $(3a+2b)^2$
3. $(ab+4c)^2$
4. $(4a+5y)^2$

5. $(105)^2=(100+5)^2$ 7. $(53)^2=(50+3)^2$

6. $(12)^2=(10+2)^2$ 8. $(27)^2=(20+7)^2$

9. $(a+b+c)^2=[a+(b+c)]^2$

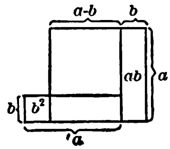
10. $(2a+1+3b)^2=[2a+(1+3b)]^2$

192. Theorem II. *The square of the difference between two quantities is the square of the first minus twice the product of the first and the second plus the square of the second quantity.*

193. Example. Multiply $(a-b)$ by $(a-b)$.

Operation:

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$



Graphically this may be shown as follows: Let a and b denote the lengths shown in the figure. Then a^2 is the large square. The figure shows that the two rectangles taken from a^2+b^2 leave $(a-b)^2$.

The formula then reads:

$$(a-b)^2=a^2-2ab+b^2$$

Exercise 62

Using the above theorem write the squares of the following expressions:

1. $(2a-b)^2$ 6. $(48)^2=(50-2)^2$
2. $(3x-2y)^2$ 7. $(59)^2=(60-1)^2$
3. $(5x-4y)^2$ 8. $(27)^2=(30-3)^2$
4. $(5-2c)^2$ 9. $(a-b+c)^2=[a-(b-c)]^2$
5. $(96)^2=(100-4)^2$ 10. $(2x-y+z)^2=[2x-(y-z)]^2$

194. Theorem III. *The product of the sum and the difference of two quantities is equal to the difference between their squares.*

Example. $(a-b)(a+b)=?$

Operation:

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

Hence $(a+b)(a-b)=a^2-b^2$.

Exercise 63

1. $(a-2b)(a+2b)$
2. $(2x-y)(2x+y)$
3. $(ac+3)(ac-3)$
4. $(2a-3c)(2a+3c)$
5. $(5-2)(5+2)$
6. $(3-1)(3+1)$
7. $(6-2)(6+2)$
8. $(10-2)(10+2)$

195. Theorem IV. *The product of two binomials having a common term is the square of the common term plus the sum of the other terms times the common term plus the product of the other terms.*

Example. $(a+b)(a+c)=?$ (a being the common term.)

Operation:

$$\begin{array}{r} a+b \\ a+c \\ \hline a^2+ab+ac+bc \\ a^2+a(b+c)+bc \end{array}$$

Hence $(a+b)(a+c)=a^2+(b+c)a+bc$.

Exercise 64

By Theorem IV write products of the following expressions:

1. $(a+d)(a+e)$
2. $(x-m)(x+n)$
3. $(2a+3)(2a-4)$
4. $(5-a)(5+b)$

5. $(3a-2)(3a+1)$
6. $(2+3)(2-1)$
7. $(3b-5c)(3b-5d)$
8. $(2x+1)(2x-5)$

196. Theorem V. *The square of the sum of any number of terms equals the sum of their squares plus twice the product of each term by each of the terms following it.*

Let the pupil show by multiplication that

1. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$
2. $(b-c+d-e)^2=b^2+c^2+d^2+e^2-2bc+2bd-2be-2cd+2ce-2de.$

Exercise 65

By Theorem V expand the following:

1. $(2a-b+3c)^2$
2. $(a+2b-2d)^2$
3. $(x-y+3z)^2$
4. $(2x+3y-z)^2$
5. $(a+b+c-d)^2$
6. $(2x+y-az+3v)^2$
7. $(2+3-1+4)^2$
8. $(4a+2b-c+3d)^2$

197. Theorem VI. *The cube of the sum of two terms is the cube of the first term, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.*

Let the pupil show by multiplication that
$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ and that $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

Exercise 66

Expand the following by Theorem VI:

1. $(x+y)^3$
2. $(x-y)^3$
3. $(2x+3y)^3$
4. $(3x-2y)^3$
5. $(5+3)^3$
6. $(4-2)^3$
7. $(4a+5b)^3$
8. $(2a-1)^3$
9. $(3+2y)^3$
10. $(ax-2b)^3$

Exercise 67

Equations Involving Squares and Products of Binomials

Solve and check:

1. $a^2+(a-2)(a+5)=(a+1)(2a-3)$
2. $(a+3)^2-(a-1)^2=3(a-4)$
3. $(7x-9)(x-3)+(x+2)^2=8(x+1)^2+20$
4. $(-r+s+1)^2-(r-s)^2=3(r-s-1)+5s$
5. $(x+2a-3)^2+12a=(x+2a)^2$
6. $(4+x)(4-x)+(x-1)(x+1)+2(x+3)^2=2x^2+31$
7. $(x-12)(x+12)-x(x+3)=2x+17$
8. $2r^2=(2r-6)(r-3)+38$
9. $(5-t)(6t+5)+(t-3)^2=10-5(t-3)^2+11t$
10. $5(1+x)(1-x)+(2x+1)^2+(x+1)(x-1)=19(x-3)$

Exercise 68—Some Simple Applications

1. Find the square of 61 by writing it $(60+1)^2$.

2. Find $(41)^2$, $(82)^2$, $(101)^2$. Check by ordinary multiplication.

3. Find the square of 19 by writing it $(20-1)^2$.

4. Find $(38)^2$, $(199)^2$, $(498)^2$. Check by ordinary multiplication.

5. Find the product of 19 and 21 by writing $(20+1)(20-1)$.

6. Find the following products: $39\cdot41$, $79\cdot81$, $48\cdot52$, $101\cdot99$.

7. A rectangle is 8 inches longer than it is wide. Express its length in terms of its width. If each dimension is increased 2 inches write the new dimensions in terms of the

old. If this increase in the dimensions makes the area 32 square inches larger, find the old dimensions.

The area of a rectangle = length times breadth, and the old area $+32 =$ the new area.

8. A rectangle is 6 feet longer than it is wide. If each dimension is increased 1 foot the area is increased 13 square feet. Find the original dimensions.

9. A rectangle is 4 feet longer than it is wide. If the dimensions are each decreased 3 feet its area is decreased 21 square feet. Find its original dimensions.

10. A rectangle is 7 feet longer than it is wide. If it were 2 feet longer and 5 feet narrower its area would be 51 square feet more. Find the dimensions.

11. A rectangle is 2 feet longer and 1 foot narrower than a certain square of equal area. Find the dimensions of both.

12. A rectangle is 8 feet longer and 8 feet narrower than a certain square of equal area. Find the dimensions of both if it is possible to do so.

If b is the base of a triangle, h the altitude, *i.e.*, the distance from the vertex to the base, and A its area, then the area equals one-half the base times the altitude or

$$A = \tfrac{1}{2}bh = \frac{bh}{2} \quad (\S\ 158)$$

13. The altitude of a triangle is 3 inches less than the base, which is 7 inches. Find the area.

14. A farmer has a triangular field with 18 acres in it. Its base is found to be 440 yards long by measurement. Does he need to measure the altitude in order to find its length? Compute it.

An acre is a square 69.57 yards on each side.

15. A triangle has an altitude twice as long as its base. Its area is decreased by 3 square feet by decreasing its height 2 feet. Find the base and altitude.

16. Compute the number of acres in a triangular field if the base is 120 yards and the altitude 211.6 yards.

Exercise 69 — Review

The exercises below all fall under some one of the six theorems of this chapter. Let the student decide which one applies, then expand. The pupil should time himself on these easy exercises. They should be done rapidly.

1. $(x+4)^2$
2. $(34)^2=(30+4)^2$
3. $(2x+1)^2$
4. $(99)^2=(100-1)^2$
5. $(x+a)(x-a)$
6. $(x+4)(x-4)$
7. $(x+4y)(x-4y)$
8. $(xy-1)(xy+1)$
9. $(xy+x)(xy-x)$
10. $(3x^2+7x)(3x^2-7x)$
11. $(101)(99)$
12. $(191)(209)$
13. $(5x-2)^2$
14. $(1-2ab)^2$
15. $(x+7)(x+3)$
16. $(y+6)(y-2)$
17. $(ab+3)(ab+7)$
18. $(xy-7)(xy-3)$
19. $(1+3x)(1-3x)$
20. $(a-7b)^2$
21. $(7c-3a)^2$
22. $(9m+3n)(9m-3n)$
23. $(x+y-a)^2$
24. $(x+y+2)^2$
25. $(x+y+2z)^2$
26. $(4ax-3ay-8)^2$
27. $(13y+8x-17)^2$
28. $(x-y)^3$
29. $(a+2)^3$
30. $(a+2x)^3$
31. $(x-2y)^3$
32. $(3x+y)^3$
33. $(6x+1)^3$
34. $(2x+3y)^3$
35. $(3x-2y)^3$
36. $(2ab+1)^3$
37. $(x-10y)^2$
38. $(a^2b-1)(a^2b+1)$
39. $(2a^2+3)(2a^2-3)$
40. $(-a+1)(a+1)$
41. $(301)^3$
42. $(99)^3$
43. $(3a^2+1)^3$
44. $(6x^3-a^2)^3$

CHAPTER X

FACTORING

DEFINITIONS

198. Factor. If an algebraic expression or algebraic number is the product of two or more other numbers each of these expressions is said to be a *factor* of the first.

For example, $6=2\times3$; hence 2 and 3 are factors of 6. $3a-3b=3(a-b)$; hence 3 and $a-b$ are factors of $3a-3b$. $a^3-ab^2=a(a-b)(a+b)$; then a, $a-b$, and $a+b$ are factors of a^3-ab^2.

199. From the definition of factor it follows that *factoring* is separating a number into other numbers such that their product is equal to the first.

200. The **square root** of an expression is one of its two equal factors.

Thus, $\sqrt{2}$ is a number such that—

$$\sqrt{2}\sqrt{2}=2,$$

and $\sqrt{a}$ is such a number that $\sqrt{a}\sqrt{a}=a$.

Finally, $\sqrt{x^2+2x+1}$ is $x+1$ since $(x+1)^2=x^2+2x+1$.

201. The **cube root** of a number is one of its three equal factors.

For example, $\sqrt[3]{2}$ is such a number that $\sqrt[3]{2}\sqrt[3]{2}\sqrt[3]{2}=2$

Similarly, $\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{a}=a$.

Finally, $\sqrt[3]{a^3+3a^2b+3ab^2+b^3}$ is $a+b$

Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

202. A **rational expression** is one which does not involve radical signs of any sort, *e.g.*, x^3+2a and $\dfrac{\frac{7}{2}x^2-2.1y+7}{16k-3}$ are rational expressions, but $\sqrt{x}+13$ and $x+\sqrt{3}$ are *irrational*.

203. An **integral expression** may have fractions in the coefficients but none of the letters representing unknowns occur in the denominator.

Thus, $6x^2-3$ and $2.5x^2+31y-\frac{4}{3}$ are integral expressions or quantities, but $\dfrac{6}{x^2}-3$ and $\dfrac{31}{y}-16a^2b+c$ are fractional expressions.

An expression may be integral in certain letters, and fractional in other letters.

Most of the expressions used in this book are rational.

FACTORING

204. The subject of factoring is very extensive. In this chapter we shall consider only rational and integral factors of rational and integral expressions. Even rational expressions do not always have rational factors. For example, x^2-2 has the factors $(x-\sqrt{2})(x+\sqrt{2})$ but these factors are excluded because they are not rational.

205. Prime Numbers. An expression which has no rational integral factor (except itself and unity), is said to be *prime*.

206. Square and Cube Roots of Monomials. The square root of 4 is one of the two equal factors of 4. But 4 may be broken up into two equal factors in two ways, viz.,

$$4=(+2)(+2) \text{ and } 4=(-2)(-2)$$

Thus it appears that 4 has two square roots, $+2$ and -2. Similarly 9 has two square roots, $+3$ and -3, since $(+3)(+3)=9$ and $(-3)(-3)=9$.

In general, every number or expression has two square roots, one positive and one negative. The positive root is called the *principal square root* and if no sign precedes the radical the positive root is always understood.

Since $x^3x^3=x^6$, $\sqrt{x^6}=x^3$

Again, since $a^4b^2\ a^4b^2=a^8b^4$, $\sqrt{a^8b^4}=a^4b^2$

These examples show that:

The exponent of any letter in the square root of a monomial is one half the exponent of the letter in the monomial.

207. By the definition of square root the two factors must be equal. This means that they must have the same sign as well as the same numerical value. Since the product of two positive numbers is positive and the product of two negative numbers is positive, it follows that a negative number has no square root in the present sense of that term. Later the meaning is extended so as to include square roots of negative numbers.

208. Since $(+2)(+2)(+2)=8$, it follows that $\sqrt[3]{8}=2$.

Similarly, since $(-3)(-3)(-3)=-27$, it follows that $\sqrt[3]{-27}=-3$.

In general: The cube root of a monomial has the same sign as the monomial itself. Since $x^2\cdot x^2\cdot x^2=x^6$, therefore $\sqrt[3]{x^6}=x^2$.

Similarly, $a^3b^2\cdot a^3b^2\cdot a^3b^2=a^9b^6$; therefore $\sqrt[3]{a^9b^6}=a^3b^2$.

Hence: the exponent of any letter in the cube root of a monomial is one third the exponent of that letter in the monomial. For convenience we collect the results deduced in the above discussion in the following rules:

209. Rule. *To extract the square root of a monomial extract the arithmetical square root of the numerical coefficient, divide*

the exponents of the letters by 2, and give the result the double sign $\pm$.

Rule. *To extract the cube root of a monomial extract the arithmetical cube root of the numerical coefficient, divide the exponents of the letters by 3, and prefix the sign of the monomial.*

Exercise 70—Square and Cube Roots

Give the following square and cube roots. Solve orally.

1. $\sqrt{9x^2}$
2. $\sqrt{4a^{\frac{7}{8}}}$
3. $\sqrt{16a^4b^2}$
4. $\sqrt{9b^6}$
5. $\sqrt{16x^8}$
6. $\sqrt{25x^{10}y^{16}}$
7. $\sqrt{49a^{12}c^{18}}$
8. $\sqrt[3]{27}$
9. $\sqrt[3]{x^6}$
10. $\sqrt[3]{-x^3}$
11. $\sqrt[3]{-a^9}$
12. $\sqrt[3]{a^6b^3}$
13. $\sqrt[3]{512x^{15}}$
14. $\sqrt[3]{-8a^3}$
15. $\sqrt[3]{-729x^6r^{12}}$
16. $\sqrt[3]{1000a^6b^6r^6}$
17. $\sqrt{x^{2n}}$
18. $\sqrt{a^{4n}}$
19. $\sqrt{a^{2n}x^2y^{2a}}$
20. $\sqrt{7^{4p}x^{2a}y^{2z}}$
21. $\sqrt[3]{x^{3n}}$
22. $\sqrt[3]{a^{3a}}$
23. $\sqrt[3]{e^{3x}}$
24. $\sqrt[3]{27e^{3ax}}$
25. $\sqrt[3]{-8\mathrm{a}^{3u}}$
26. $\sqrt[3]{8x^{12n}y^{3a}}$

POLYNOMIALS WITH A COMMON MONOMIAL FACTOR

210. If a common monomial factor occurs in each term of a polynomial the polynomial can be factored by dividing each of its terms by the common factor and writing the quotients inside a parenthesis, with the common factor outside, thus:

$$ax^2+ba+ac=a(x^2+b+c)$$

211. It will be helpful for the student to review §§ 179 and 180.

To factor $8a^2b-24b^3$.

The common factor is $8b$.

Hence $8a^2b-24b^3=8b(a^2-3b^2)$.

Performing the multiplication gives a check.

Exercise 71—Monomial Factors

Factor and check:

1. $2a+6$
2. $a^2+2ab+3a$
3. $3x^3-15x^6$
4. $11a^4b-18a^3b^4-3a^3b^3$
5. c^3-c-c^2
6. $14xy^4-21x^3y+x^4y^4$
7. $8a^2-15a+6a^3$
8. $3x^3y^4-18x^4y^3+21x^2y^3$
9. $6t^6+10t^{10}-20t^2$
10. $72x^3y^4z^2+36x^4y^3z-48x^3y^3z^4$
11. $-2a^2x^2y^2-3a^5x^2y^3-6a^6x^5y^3z^2+a^4x^4y^4z$
12. $6x^{2a}y+x^ay^2$
13. $3a^{4n}b^{3n}+a^{6n}b^{3n}-a^{5n}b^{5n}$

POLYNOMIALS FACTORED BY GROUPING TERMS

212. Typical Form. The typical form is

$$ax+ay+bx+by$$

Factoring a out of the first two terms and b out of the second two we have

$$ax+ay+bx+by=a(x+y)+b(x+y)$$

This shows the common factor $x+y$. Therefore, we have $(x+y)(a+b)$.

Polynomials which can be factored by the foregoing process usually contain four, six, or eight terms. *Notice carefully* that an expression is not factored when its separate terms are factored. For example, $49a^4+21a^3-7a^2+7a$ is not

factored by writing $7a^3(7a+3)-7a(a-1)$, but by writing $7a(7a^3+3a^2-a+1)$.

The success of this method depends on the student's ability to group the terms so as to show a common factor. Sometimes several arrangements may be tried before a successful one is found.

Exercise 72

Factor by grouping terms. Check. See how quickly you can solve and check the first six.

1. $ax^2+ay^2-dx^2-dy^2$

Solution: $ax^2+ay^2-dx^2-dy^2=a(x^2+y^2)-d(x^2+y^2)$
$=(x^2+y^2)(a-d)$.

2. $3(x+y)+a(x+y)$
3. $ax-by+bx-ay$
4. $x(x-a)-a(x-a)$
5. $kr-ks+3s-3r$
6. $2y^2-ay-2xy+ax$

Sometimes the parentheses must be removed and the terms grouped in a different order before a common factor of all the terms appears.

7. $x(a-b)+3(b-a)$

Removing the parentheses
$ax-bx+3b-3a=ax-3a-bx+3b=$
$a(x-3)-b(x-3)=(x-3)(a-b)$.

8. $x(a-2y)+3(2y-a)$
9. $x^2(a-b)+y^2(b-a)$
10. $5(3x-4ax)+2(4a-3)$
11. $a(1-a)+(1-a^2)$

Notice that two apparently different sets of factors may be obtained, as in the following example:

12. $ax-2bx-ay+2by$

$ax-2bx-ay+2by=x(a-2b)-y(a-2b)=(a-2b)(x-y)$.
Second solution: $ax-2bx-ay+2by=2by-2bx-ay+ax=$
$2b(y-x)-a(y-x)=(y-x)(2b-a)$.

The difference, however, is only in the sign of each factor and arises from the fact that the product of two negative quantities is a positive quantity.

13. $2ar+2as+4br+4bs$

14. $ax^2-3bxy-axy+3by^2$

15. $4xy-3mn+2xn-6my$

16. $4a^3-10a^2b-6a+15b$

17. $-2ax+7a^2+16bx-56ab$

18. $15h^3+30h-h^2-2$

PERFECT SQUARES

213. In Chapter IX we studied the identities

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

A trinomial is said to be a perfect square if it is the square of some binomial. A trinomial is a perfect square if

1. Two terms are squares of monomials.

2. The other term is twice the product of these monomials with either $+$ or $-$ sign.

For example, $a^2+2ab+b^2$ is a perfect square and it fulfills the two specified conditions, for (1) a^2 and b^2 are squares of the monomials a and b, and (2) the middle term is twice the product of a and b.

Again, $9x^2-30ax+25a^2$ has the first and last terms squares of the monomials $3x$ and $5a$, while the middle term is $-2(3x)(5a)$. Therefore $9x^2-30ax+25a^2=(3x-5a)(3x-5a)=(3x-5a)^2$.

One of the most common processes in algebra is to supply the third term of a trinomial so that the three may form a perfect square.

Exercise 73

Supply the missing terms so that the three form a perfect square:

1. $4y^2+36y+?$

The square root of the first term $\sqrt{4y^2}=2y$. The missing term is the square of some number T such that $2(2y)T=36y$ or $4yT=36y$. Therefore $T=9$ and $4y^2+36y+81=(2y+9)(2y+9)$.

Multiplication checks the result.

2. $x^6-(?)+9$

The missing term is twice the product of the monomials whose squares produce the end terms, that is, $2(x^3)3=6x^3$. Therefore

$$x^6-6x^3+9=(x^3-3)(x^3-3).$$

Multiplication again gives the check.

3. $c^2-(?)+d^2$
4. $x^2+(?)+1$
5. $a^4+(?)+4$
6. $4c^2+(?)+1$
7. $25x^2+(?)+36a^2$
8. $x^2+2x+(?)$
9. $4y^2+16y+(?)$
10. $16t^4-16st^2+(?)$
11. $64h^6-32ah^3+(?)$
12. $121a^8-88a^4x^5+(?)$
13. $4x^2+(?)+25$
14. $4a^{16}+(?)+169k^2$

Test the following as to whether they are squares. Factor them if they are, and if they are not already squares modify them to make them squares, and factor the modified forms.

15. $x^2-2xy+y^2$
16. $x^4+2x^2y^2+y^4$
17. $x^4+2x^2y^3+y^6$
18. x^4+1+2x^2
19. $4a^2+2mn+n^2$
20. $100+x^2+20x$
21. $169a^4-150a^2x^4y^3+36x^8y^6$
22. $121a^2+4a^2b^4-22ab^2$

THE DIFFERENCE OF TWO SQUARES

In § 194 we found by multiplication

$$(a+b)(a-b)=a^2-b^2$$

From this we get the following rule:

214. Rule. *To factor the difference of two squares extract the square root of each term, add these roots for one factor, and subtract the second from the first for the second factor.*

For example, $4-(a+b)^2$.

$4-(a+b)^2=[2+(a+b)]\,[2-(a+b)]=(2+a+b)(2-a-b)$.

Sometimes the factors obtained by this rule are not prime. In that case the student is expected to factor again till each factor is prime.

For example: x^4-1.
$x^4-1=(x^2+1)(x^2-1)$.
But $x^2-1=(x+1)(x-1)$.
Therefore $(x^4-1)=(x^2+1)(x+1)(x-1)$.

Exercise 74

Factor and check by multiplication. Solve and check ten of these in ten minutes.

1. x^2-y^2
2. x^2-4y^2
3. k^2-1
4. x^4-a^2
5. $16-c^2$
6. $9a^4-36x^2$
7. $16a^2-25x^2$
8. a^2b^4-169
9. $1-16a^8$
10. x^8-y^8
11. $121x^2b^2-1$
12. $144a^4b^6-c^2$
13. $1-(a-b)^2$
14. $(a+b)^2-9$
15. $16-(2x-y)^2$
16. $(a-3b)^2-4x^2$
17. $16(2b-a)^2-(a+b)^2$
18. $a^2-2ab+b^2-c^2$
19. $x^2+4xy+4y^2-y^2$
20. $a^2+4ab+3b^2$
21. $x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2=(x^2+y^2)^2-(xy)^2=(x^2+y^2-xy)(x^2+y^2+xy)$

It is important to know whether a given expression can be written as the difference of two squares. Make the necessary corrections below, then factor.

22. $9x^2-a^2+2ab-b^2$
23. $a^2-x^2-4xy-6y^2$
24. $x^2-a^2+y^2-4-2xy+4a$
25. $a^2+2ab+b^2-c^2-d^2-2cd$
26. $a^2+4ab+b^2-x^2$
27. $1+2b-c^2+2bc-b^2+a^2$
28. $c^2-(a^2-2ab+b^2)$
29. $16x^2-y^2+14yz-49z^2$

TRINOMIALS OF THE FORM x^2+ax+b.

215. If x^2+ax+b can be factored into rational factors they must be binomials whose first term is x; thus

$$x^2+ax+b=(x+h)(x+k)$$

Multiplying the numbers on the right hand side,

$$x^2+ax+b=x^2+(h+k)x+hk$$

When an expression is factored, only its form is changed. The equal signs in factoring problems are all signs of *identity.* If the equation above is an identity the right member must be the same number as the left member only in different form. This means that the coefficients of like powers of x must be equal.

Therefore, $a=h+k$ and $b=hk$

Hence, in order to factor an expression like x^2+ax+b we have the following rule:

216. Rule. *Find two numbers whose sum is **a** and whose product is **b**. Add each of them to the square root of the first term for the factors.*

Examples

1. $x^2+3x+2=(x+2)(x+1)$.

This is true, since the sum, $2+1$, is the coefficient of x, and the product, $2\cdot1$, is the term independent of x. This factoring can usually be done by inspection. If such numbers do not exist, then the trinomial cannot be factored. Many trinomials of the form x^2+ax+b have no rational factors.

2. $x^2-11x+24$.

Here it is necessary to find two numbers whose product is $+24$ and whose sum is -11. Evidently -8 and -3 fill these conditions. Hence,

$$x^2-11x+24=(x-8)(x-3)$$

3. x^2-x-2.

The two numbers to be found must have unlike signs since their product is -2. The larger must be negative since their sum is -1. Therefore the numbers are -2 and $+1$. Therefore $x^2-x-2=(x-2)(x+1)$.

4. x^2-2x+2.

Here the two numbers to be found must be such that their product is 2 and their sum is -2. Clearly no such numbers exist and this expression has no rational factors.

Exercise 75

Factor and check by multiplication. Solve and check all odd numbered exercises in thirty minutes.

1. x^2-3x+2

2. x^2+6x+5

3. x^2-x-6

4. x^2+x-6

5. $a^2-19x+18$

6. $a^2+2x-35$

7. a^2+x-30

8. z^2+7z+6

9. $t^2+11x-42$

10. t^2+x-42

11. t^2-x-42

12. $3+2x-x^2$

Solution: $3+2x-x^2=(3-x)(1+x)$.

13. $15+2c-c^2$

14. $54-15x-x^2$

15. $95+14x-x^2$

16. $x^2+8ax+12a^2$

17. $x^2y^2+11xy-210$

18. $a^2x^2-6axy-27y^2$

19. $p^2+3p-154$

20. $x^2+(a+b)x+ab$

21. $h^2k^2+10hk^2-24k^2$

22. a^6-11a^3+18

23. $a^2x^6-3ax^3y-28y^2$

24. $(x-y)^2-14(x-y)+40$

25. $9a^2x^4y^6+a^4x^8-22y^{12}$

26. $a^6x^2-a^3b^2xy^4-30b^4y^8$

TRINOMIALS OF THE FORM ax^2+bx+c

217. The method of factoring such expressions as ax^2+bx+c is perhaps best illustrated by special cases.

Examples

1. To factor $3x^2+5x+2$.

The factors are evidently binomials. The product of the end terms of these binomials must be $3x^2$ and 2.

Hence the binomials may be either

$$\begin{array}{rl} & (3x+1)(x+2) \\ \text{or} & (3x+2)(x+1) \end{array}$$

The first pair, when multiplied together, give $7x$ for the middle term. Therefore these factors are rejected. The second pair give $5x$ for the middle term. Therefore the factors are $(3x+2)$ and $(x+1)$.

2. To factor $12h^2-11h+2$.

The end terms of the factors must be chosen so that their product gives $12h^2$ and 2. There are several ways of doing this. It is helpful to arrange the tentative factors as if for multiplication, as follows:

$$\begin{array}{ccccc} 12h-1 & 6h-1 & 6h-2 & 4h-2 & 4h-1 \\ h-2 & 2h-2 & h-1 & 3h-1 & 3h-2 \\ \hline -25h & -14h & -8h & -10h & -11h \end{array}$$

The second sign must be minus, since the last term is $+$ and the middle term $-$. The cross-products are indicated above for convenience. From these it appears that the factors are $(4h-1)(3h-2)$. Why?

218. After a little practice one can shorten this work. For instance, one can discard the factors $(6h-1)$ $(2h-2)$ because $2h-2$ has the factor 2, which is not a factor of the original trinomial and therefore cannot be contained in any of its

factors. For the same reason the tentative factors $(6h-2)$ $(h-1)$ and $(4h-2)$ $(3h-1)$ may be discarded.

219. The following arrangement of the work is suggested.

To factor $9x^2+18x+8$.

The trial factors are:

$$\begin{array}{ccccc} 9x+1 & 9x+8 & 3x+4 & 3x+8 & 9x+4 \\ \times & \times & \times & \times & \times \\ x+8 & x+1 & 3x+2 & 3x+1 & x+2 \\ \hline +73x & +17x & +18x & +27x & +22x \end{array}$$

Therefore $9x^2+18x+8=(3x+4)(3x+2)$.

Check:
$$\begin{array}{r} 3x+4 \\ 3x+2 \\ \hline 9x^2+12x \quad\quad \\ +\ 6x+8 \\ \hline 9x^2+18x+8 \end{array}$$

Exercise 76

Factor every third one of the following list in twenty minutes.

1. $3a^2+5a+2$
2. $2a^2+5a+3$
3. $9x^2+9x+2$
4. $2b^2-7b+3$
5. $9c^2+36c+32$
6. $6x^2+7x-3$
7. $5t^2-8t+3$
8. $8x^2+18x+9$
9. $10a^2-29a+10$
10. $12y^2-y-6$
11. $14y^2+59y-18$
12. $8x^2-14x-15$
13. $5a^2+18ab+16b^2$
14. $5a^2b^2+18ab+16$
15. $7x^4+123x^2-54$
16. $21t^4-t^2-2$
17. $10-5c^2-15c^4$
18. $3+7x^2-6x^4$
19. $20r^6-9r^3-20$
20. $6+9x-42x^2$

21. $5a^2+5ax-10x^2$

22. $6a+9ax-42ax^2$

23. $5x^4y^2b^6+9x^2yb^3c^2-18c^4$

24. $36a^3d^2+24a^3d-12a^3$

DIVISION BY A BINOMIAL

220. The division of a polynomial by a monomial was discussed in §§ 183 to 184. It is fairly easy to extend our knowledge to the case where the divisor is a binomial provided the dividend contains the divisor as a factor.

For example, divide x^2+2x+1 by $x+1$.

$$\frac{x^2+3x+2}{x+1}=\frac{(x+1)(x+2)}{x+1}=x+2$$

This example makes clear the method. In case the dividend cannot be factored in such a way that the divisor cancels with one of its factors we may proceed as in the following:

For example, divide x^2+3x+5 by $x+1$.

$$\frac{x^2+3x+5}{x+1}=\frac{x^2+3x+2+3}{x+1}=\frac{(x+1)(x+2)}{x+1}+\frac{3}{x+1}=x+2+\frac{3}{x+1}$$

There is a remainder 3 which cannot be divided by $x+1$. The result therefore consists of a quotient plus a remainder. The divisor must always be written under the remainder as was done in arithmetic. The student will help his understanding of this process by reviewing long division in arithmetic.

It is sometimes necessary to know whether a given binomial is a factor of a trinomial. Such a question can be answered by performing the division as above.

221. In all division problems hereafter the following important relation holds:

$$\frac{\textit{dividend}}{\textit{divisor}}=\textit{quotient}+\frac{\textit{remainder}}{\textit{divisor}}$$

Exercise 77

Perform the following divisions by factoring. Try to solve these mentally.

1. $\dfrac{x^2+4x+3}{x+3}$

2. $\dfrac{x^2+4x+7}{x+3}$

3. $\dfrac{x^2-5x+6}{x-2}$

4. $\dfrac{x^2-5x+14}{x-2}$

5. $\dfrac{x^2-x-42}{x+6}$

6. $\dfrac{x^2-x+42}{x-7}$

7. $\dfrac{2x^2+3x-1}{x-1}$

8. $\dfrac{3x^2+3x-9}{3x-3}$

9. $\dfrac{6x^2-5x+7}{2x-1}$

10. $\dfrac{4x^2+10x+4}{2(x+2)}$

DIVISION BY A POLYNOMIAL

222. In case the divisor is a polynomial the method is more complicated. It is very similar to long division in arithmetic. Example:

Dividend =	$3x^4-x^3+4x^2-5x+16$	x^2+x-6 divisor
1st product		$3x^2-4x+26$
$3x^2(x^2+x-6)$	$=3x^4+3x^3-18x^2$	1st quotient, 2d quotient, 3d quotient
By subtraction	$-4x^3+22x^2-5x+16$	
2d product		
$-4x(x^2+x-6)=$	$-4x^3-4x^2+24x$	
By subtraction	$26x^2-29x+16$	
3d product		
$26(x^2+x-6)=$	$26x^2+26x-156$	
	$-55x+172$ remainder	

Therefore $\dfrac{3x^4-x^3+4x^2-5x+16}{x^2+x-6}=3x^2-4x+26+\dfrac{-55x+172}{x^2+x-6}$

Check. Let $x=1$.

$$\frac{3-1+4-5+16}{1+1-6}=3-4+26+\frac{-55+172}{1+1-6}\cdot$$

$$\frac{17}{-4}=25+\frac{117}{-4}$$

$$\frac{17}{-4}=\frac{17}{-4}, \text{ check.}$$

223. Explanation. 1. Arrange the dividend and divisor according to ascending or descending powers of some letter. In the above both are arranged in descending powers of x.

2. Obtain the first term of the quotient by dividing the first term of the dividend by the first term of the divisor, *i.e.*, $3x^4 \div x^2 = 3x^2$.

3. Multiply the divisor by this partial product, placing the result under the dividend, and subtracting.

4. Divide the first term of this remainder by the first term of the divisor and proceed as before till there is a remainder of lower degree than the divisor. Then arrange as follows:

$$\frac{dividend}{divisor}=quotient+\frac{remainder}{divisor}$$

5. Check by substituting numerical values for the letters, being careful to avoid values that make the divisor vanish.

224. If the remainder is zero then the divisor is a factor of the dividend. The principal value of long division is the aid it gives in finding the factors of complicated polynomials.

In the problems below note carefully those cases in which the divisor is a factor of the dividend.

Exercise 78

Perform the indicated divisions and check. See how many of these you can solve without use of pencil and paper.

1. $\dfrac{x^2+2xy+y^2}{x+y}$ 2. $\dfrac{x^3+3x^2+3x+1}{x+1}$

3. $\dfrac{x^3-3x^2+3x-1}{x-1}$

4. $\dfrac{a^3+3a^2b+b^3+3a^2b}{a+b}$

5. $\dfrac{2x^3+x^2-x-1}{x+3}$

6. $\dfrac{3x^3+7x^2-4x}{x-2}$

7. $\dfrac{x^3+4x^2+x-6}{x-3}$

8. $\dfrac{4x^4+3x^3+7x-16}{2x+3}$

9. $\dfrac{4x^4+3x^3y+7xy^3-16y^4}{2x+3y}$

10. $\dfrac{x^4+2x^3-6x^2-31x+6}{x^2-x+2}$

11. $\dfrac{x^4-1}{x^2-1}$

12. $\dfrac{a^4+a^2+1}{a^2-a+1}$

13. $\dfrac{x^3-8x^2+75}{x-5}$

14. $\dfrac{x^3+2y^3+3z^3-4xyz}{x+y+z}$

15. $\dfrac{a^5+5a^4+10a^3+10a^2+5a+1}{a^2+2a+1}$

16. $\dfrac{x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5}{x^2+2xy+y^2}$

17. $\dfrac{a^5+1+10a^2+10a^3+5a^4}{a^2+1+2a}$

18. $\dfrac{x^3-y^3}{x-y}$

19. $\dfrac{x^3+y^3}{x+y}$

20. $\dfrac{x^4-y^4}{x^2-y^2}$

21. $\dfrac{x^4-y^4}{x-y}$

22. $\dfrac{x^4-y^4}{x^3-y^3}$

23. $\dfrac{x^4-y^4}{x+y}$

24. $\dfrac{x^4+y^4}{x^2+y^2}$

25. $\dfrac{x^3-y^3}{x+y}$

26. $\dfrac{x^3+y^3}{x-y}$

27. $\dfrac{a^2+ab+b^2}{a+b}$

28. $\dfrac{x^4+x^3y+xy^3+y^4}{x+y}$

30. $\dfrac{x^3+3xy+y^3-1}{x+y-1}$

29. $\dfrac{x^5+x^4y+x^3-x^3y^2+y^3-2ay^2}{x^2+xy-y^2}$

31. $\dfrac{x^{3n}+3x^ny^n+y^{3n}-1}{x^n+y^n-1}$

32. Make a list of those problems in which the divisor is a factor of the dividend.

FACTORS OF THE SUM AND DIFFERENCE OF TWO CUBES

225. There are two examples in the last set of exercises which deserve particular attention— numbers 18 and 19. The results of the divisions required are

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2 \text{ and } \frac{x^3-y^3}{x-y}=x^2+xy+y^2$$

By these two identities we can factor any expression which can be written as the sum or difference of two cubes.

Example: Factor $8x^3+1$.

Solution: $8x^3+1=(2x)^3+1=(2x+1)\left\{(2x)^2-(2x)+1\right\}=(2x+1)(4x^2-2x+1)$.

Example: Factor $27(a+b)^3-8c^3$.

Solution: $27(a+b)^3-8c^3=3^3(a+b)^3-(2c)^3=$
$\left\{3(a+b)-2c\right\}\left\{3^2(a+b)^2+3(a+b)(2c)+(2c)^2\right\}=$
$(3a+3b-2c)\left\{9(a+b)^2+6c(a+b)+4c^2\right\}$.

Check: If $a=1$, $b=1$, $c=1$.
$27(1+1)^3-8=(3+3-2)\left\{9\cdot4+6\cdot2+4\right\}$
$208=4\cdot52$; $208=208$.

The above methods are summarized as follows:

226. Rules. *The sum of two cubes is always divisible by the sum of the numbers. The quotient is the sum of the squares of the numbers minus their product.*

The difference of two cubes is always divisible by the difference of the numbers. The quotient is the sum of the squares plus their product.

Exercise 79

Factor and check. Try to solve first nine in ten minutes.

1. a^3+1	**4.** $27x^3+1$	**7.** a^3+b^6
2. a^3-1	**5.** $64-a^3$	**8.** y^9-x^3
3. $27x^3-1$	**6.** $8x^3-y^3$	**9.** a^6-64

Solution: $a^6-64=(a^3)^2-8^2=(a^3-8)(a^3+8)=(a-2)(a^2+2a+4)(a+2)(a^2-2a+4)$.

Check: $a=1$. $1-64=(1-2)(1+2+4)(1+2)(1-2+4)$

$$-63=(-1)(7)(3)(3)$$
$$-63=-63.$$

Again, a^6-64 may be regarded as the difference of two cubes directly, as: $a^6-64=(a^2)^3-(4)^3=(a^2-4)(a^4+4a^2+16)$

$$=(a-2)(a+2)(a^4+4a^2+16).$$

Check: $a=1$. $1-64=(1-2)(1+2)(1+4+16)$

$$-63=(-1)(3)(21)$$
$$-63=-63$$

In the last solution no general method was known for breaking a^4+4a^2+16 into its factors. Division, however, will show that it has the factors $(a^2+2a+4)(a^2-2a+4)$.

Or we may factor it by putting $a^4+4a^2+16=a^4+8a^2+16-4a^2$ $=(a^2+4)^2-(2a)^2=(a^2+4-2a)(a^2+4+2a)$.

10. $27t^3-125s^3$	**14.** $8x^9r^6-125t^3$
11. x^3y^3+1	**15.** a^6-c^{12}
12. $a^3x^6+y^3$	**16.** $729a^3+8b^6$
13. $729-a^6$	**17.** $125x^9-27y^{12}$

SUMMARY OF FACTORING

227. In the previous articles the various cases of factoring have been treated separately. Often, however, several cases will be applied in one exercise. The pupil who does not recognize the standard type forms in most of the following *exercises* has not made sufficient progress to enable him to

proceed successfully. *Only the most important* cases of factoring have been considered and they *must be thoroughly mastered.*

SUGGESTIONS AS TO METHOD OF ATTACK

I. Examine the expression for a common monomial factor, and if one is found, factor into the monomial times a polynomial.

II. Examine the polynomial to determine which case it is like. For convenience the *type forms** are here collected and named:

1. $ax+ay+bx+by=a(x+y)+b(x+y)=(a+b)(x+y)$, grouping terms.
2. $x^2\pm 2xy+y^2=(x\pm y)^2$, perfect square.
3. x^2+bx+c, a trinomial with first coefficient 1.
4. ax^2+bx+c, the general trinomial.
5. $a^2-b^2=(a-b)(a+b)$, the difference of two squares.
6. $a^2+2ab+b^2-c^2=(a+b)^2-c^2$, again the difference of squares.
7. $x^3\pm y^3=(x\pm y)(x^2\mp xy+y^2)$, the sum or difference of cubes.

III. After I and II have been applied examine each factor to see whether or not it can be factored again.

IV. Check by multiplication or numerical substitution.

Exercise 80

See how many of the following you can solve mentally:

1. x^2-5x+6
2. $a^2+16ab+64b^2$
3. $1-a^2$
4. $3x^3-75x$

* By "type forms" is meant a statement of a theorem in symbols.

5. $a^2+17ab+30b^2$
6. $8-27x^3$
7. $x^2+10xy+9y^2$
8. $x^2-10xy+9y^2$
9. $3x^3+x^2-2x$
10. $9a^2-b^2$
11. $4x^4-20x^2y^2+9y^4$
12. $2a^3b^3+8ab-8a^2b^2$
13. $ax-2ay+6bx-12by$
14. $3x^3-45x^2+168x$
15. $2r^2-6rs+9st-3rt$
16. $x^2-2xy-2xz+4yz$
17. $3a^2+12b^4+12ab^2$
18. $x^4-25y^2+10x^2y-z^2$
19. $p^4+q^4+2p^2q^2$
20. $64x^6-x^6y^3$
21. $3cx-6c+dx-2d$
22. $1+a^3$
23. $(a+1)^2-(b+1)^2$
24. $(a+b)^2-a^2$
25. $(c-d)^2-(c-d)^2$
26. $2a^2x^2+34ax^3+144x^4$
27. $2x^2y+3xy^2+y^3$
28. $3x^4y-3x^2y^2-36y^3$
29. $10-y^2+10y^3-y^5$
30. $9x-6xy^2-x^9+xy^4$
31. $2000-2a^6$
32. $6rs+12rb-30ys-60yb$
33. $27(a+1)^3-8(a-1)^3$
34. $15y(x^2-1)+224xy$
35. $576-(18-9x^2)^2$
36. $144+2a^2b-a^4-b^2$
37. $x^3y^6+y^9$
38. a^4-5a^2-84

Exercise 81 — Review

Factor these numbers by any of the methods of this chapter.

1. $x^2-kx+21x-21k$
2. a^2-y^2+2y-1
3. $(x-1)(2x-3)-6(x-1)$
4. a^9-b^9
5. $2(a^3-1)+7(a^2-1)$
6. $ax+ay+bx+by$
7. $a^2+2ab+b^2$
8. $3x^2-5x+2$
9. x^9-8
10. m^4n-mn^4

11. $1-49x^2$
12. x^3y^3+1
13. $x^2-11x+18$
14. $a^2x^2-21ax+90$
15. $x^2-22x+121$
16. m^3+27n^3
17. $ab-by-a+y$
18. x^6+64b^6
19. $3x^8-51x^4+48$
20. $1-a^2+2ab-b^2$
21. $(x-5y)^2-4b^2$
22. x^5+x^4+4x+4
23. $343-x^3$
24. $125x^6+27y^{15}$
25. x^4-81
26. P^2-P-72
27. $x^5-x^4-16x+16$
28. $x^2-25xy+150y^2$
29. $25-10x+x^2$
30. $3a^2+8a+4$
31. $a^2+\frac{2}{b}+\frac{1}{b^2}$
32. x^3-3x^2-x+3
33. a^3-ab^2
34. $(a+b+c)^2-1$
35. $x^2+(a-b)x-ab$
36. x^5-8x^3+16x
37. $7a^3-7$
38. $2-x-x^2$
39. $x^2+2x+1-4(x+1)$
40. $a^2-4ab+3b^2-6a+6b$
41. $(x+4)(x+3)-18(x+3)$
42. $p^3-5p^2+2p-10$
43. $3a^4-3a^2-36$
44. 1001
45. $x^3-y^3+x^2-y^2$
46. a^5+4a^3-5a
47. $a^{6n}-b^{3n}$
48. $a^2-2ab+b^2-a+b$
49. 64001
50. a^5-16a
51. $16-a^2-b^2+2ab$
52. $x^2+16x+63$
53. $a^6x^2-5a^3x-14$
54. $250x-2x^7$
55. $5xy-10y-3x+6$
56. $x^2+7x-44$
57. $a^6-a^5x-ap+px$
58. 992
59. $64x^6-729y^6$
60. $a^2+2ab+3ac+6bc$

CHAPTER XI

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

HIGHEST COMMON FACTOR

228. If a factor is contained in two or more expressions it is said to be a *common factor* of those expressions.

The common factor of several numbers which is of highest degree and which has the largest numerical factor is known as the *highest common factor* (h. c. f.) of those numbers.

Evidently the h. c. f. is the product of the prime factors common to the group of members.

229. Method of Finding H. C. F. Consequently, one method of finding the h. c. f. of several quantities is to determine the prime factors of the quantities, and multiply together those factors common to the quantities. The product will be their h. c. f.

For example, find h. c. f. of a^4-b^4, a^3+b^3, and a^2-b^2.

$$a^4-b^4=(a-b)(a+b)(a^2+b^2)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^2-b^2=(a+b)(a-b)$$

The only factor common to all the quantities is $a+b$; hence $a+b$ is the h. c. f. of the three given expressions.

Exercise 82

Find the h. c. f. of each of the following groups of expressions:

1. $a^2+5a+4,\ a^2+2a-8,\ a^2+7a+12$
2. $x^2-3x+2,\ x^2+5x-6,\ x^2-4x+3$
3. $2x^2+7x+5,\ 6x^2+5x-1$

4. $x^2-3x-18,\ 2x^2-x-21,\ 3x^2+4x-15$

5. $x^3+3x+2,\ x^2-4,\ x^2+4x+4$

6. $2x^2-7x+6,\ 4x^2-11x+6$

LOWEST COMMON MULTIPLE

230. Definition. When one expression contains another expression as a factor, the first expression is said to be a *multiple* of the second. When an expression is a multiple of each of several expressions it is called a *common multiple* of those expressions, and when it is their common multiple of lowest degree and of least numerical factor it is called the *lowest common multiple* (l. c. m.).

Examples: 1. What is the l. c. m. of $8a^2b^3$, $6ab^2$, $10a^3b$?

$$8a^2b^3 = 2^3a^2b^3$$
$$6ab^2 = 2\cdot 3ab^2$$
$$10a^3b = 2\cdot 5a^3b$$
$$\text{l. c. m.} = 3\cdot 2^3\cdot 5\cdot a^3b^3 = 120a^3b^3$$

2. Find l. c. m. of a^2-b^2, $a^2-2ab+b^2$, $4(a+b)$

$$a^2-b^2 = (a+b)(a-b)$$
$$a^2-2ab+b^2 = (a-b)^2$$
$$4(a+b) = 2^2(a+b)$$
$$\text{l. c. m.} = 2^2(a-b)^2(a+b)$$

The l. c. m. of several quantities can be found thus:

231. Rule. *Separate each quantity into its prime factors; take each factor the greatest number of times it occurs in any one quantity; the product of these factors is the l. c. m. of the quantities.*

Exercise 83

Find l. c. m. of the following groups of expressions:

1. $2a+2,\ a^2-2a-3$
2. $5a-a^2,\ 10+3a-a^2$
3. $a^2-3a-54,\ a^2-10a+9$
4. $5x^2+7x-6,\ x^2-15x-3x$
5. $x^3+3x+2,\ x^2-4,\ x^2-1$
6. $2x^2-7x+6,\ 4x^2-11x+6$

232. Summary of Chapter XI. In this brief chapter have been given the simplest methods for determining the h. c. f. and the l. c. m. of groups of algebraic numbers.

The methods are given in the following rules:

Rule: To find the h. c. f. of several numbers resolve each of them into its prime factors. The product of those factors common to all the numbers is their h. c. f.

Rule: The l. c. m. of a set of numbers must contain *all* the factors of the several numbers, hence the l. c. m. is the product of all the factors occurring in these numbers, each taken the greatest number of times it occurs in any one of the numbers.

The methods discussed in this chapter are very important. The pupil will be quite helpless in much that follows unless he is complete master of these methods of finding the h. c. f. and the l. c. m.

Exercise 84

Find h. c. f. of the following sets of numbers:

1. $x^2+18x+77,\ x^2+22x+121,\ x^2+x-110$
2. $a^3-27,\ a^2-6a+9,\ ab-3b+a-3$
3. $a^2+5a+4,\ a^2+2a-8,\ a^2+7a+12$
4. $5x^2-10x+5,\ 25x-25,\ x^2-1$
5. $x^2-3x-18,\ 2x^2-x-21,\ 3x^2+4x-15$
6. $m^2+7m+12,\ mn+3n+2m+6,\ m^2-9$
7. $b^2-9b-10,\ b^2-7b-30$
8. $2a^2+a-1,\ 4a^2-1,\ 2a^2+3a-2$

Find l. c. m. of the following sets of numbers:

9. $a^2-3a-54,\ a^2-10a+9$
10. $a-1,\ a^2-1,\ a-2,\ a^2-4$

11. x^2-y^2, $(x-y)^2$, x^3-y^3

12. $2a^2+a-1$, $4a^2-1$, $2a^2+3a+1$

13. c^2+c-12, $c^2-13c+36$, c^2-16

14. a^4-b^4, $a^3+a^2b+ab^2+b^3$

15. $b^2-9b-10$, $b^2-7b-30$

16. $a^2-3a-40$, $3a^2+15a$

Find h. c. f. and l. c. m. of these sets of numbers:

17. $(ax+ay)^2$, $a(x^2-y^2)$, ab^2y

18. (a^3+b^3), $c(a+b)^2$, a^2-b^2

19. x^3+x^2-x-1, $3(x^2-1)$

20. $3a^2-12ab-15b^2$, $4(a^2-10ab+25b^2)$

21. $2a^2c+12abc+18b^2c$, $4a^2-36b^2$

22. $x^3y^3-4ax^2y^2+4a^2xy$, $3b^2x^2y^2-12a^2b^2$

CHAPTER XII

ALGEBRAIC FRACTIONS

DEFINITIONS

233. Arithmetical fractions have been studied in Chapter I. The student is urged to review the fundamental principles and rules of operation derived there. This is necessary because every rule derived there is applicable here. The student who has a thorough knowledge of arithmetical fractions rarely has much difficulty with algebraic fractions.

234. In arithmetic we defined a fraction as one or more of the equal parts of a unit. This definition is not broad enough for our present purposes. Indeed, it failed to cover such fractions as $\frac{3}{2\frac{1}{2}}$. We now take up a more formal study of fractions and give the definition:

A fraction is an indicated quotient.

For example, $\frac{a}{b}$, $\frac{1}{1-a}$, $\frac{4x^2-21y}{3x^3-4x-a^2}$ are fractions.

The definitions of numerator, denominator, terms of a fraction, complex fractions, given in Chapter I, need not be repeated here since there is no change needed in them.

235. Like the fractions met in Chapter I, *algebraic fractions* may be proper or improper, but not in the same sense. The reader will remember that in Chapter I a *proper fraction* was defined as a fraction in which the numerator was less than the denominator, and an *improper fraction* as a fraction in which the numerator was equal to or greater than the denominator.

236. Proper and Improper Algebraic Fractions. A *proper algebraic fraction* is a fraction whose numerator is of lower degree (*i.e.*, of lower *power* or *exponent* in the same letter) than the denominator; while an *improper algebraic fraction* is a fraction whose numerator is of equal or higher degree (*i.e.*, of equal or higher power in the same letter) than its denominator.

Examples: $\frac{a^2-1}{a^3+2}$, $\frac{a}{b}$, $\frac{c^3+bc+2}{c^5-3c-1}$ are proper algebraic fractions.

$\frac{a^2-1}{a+2}$, $\frac{ab^2+1}{b}$, $\frac{c^4+5c-1}{c^2+2c-7}$ are improper algebraic fractions.

237. Two fractions that have the same value but are of different form are said to be *equivalent fractions.*

Examples. $\frac{1}{2}$ and $\frac{2}{4}$, $\frac{a^2}{ab}$ and $\frac{a}{b}$, $\frac{x-1}{x^2-1}$ and $\frac{1}{x+1}$ are equivalent fractions.

238. Changing the form of a fraction without changing its value is called *reduction of fractions.*

239. A fraction made up of an integer joined to a fraction by a plus or minus sign is known as a *mixed number.*

240. An improper fraction can be reduced to a mixed number by dividing the numerator by the denominator and adding to or subtracting from the part of the quotient thus found, the remainder being written as a fraction.

Example. Change to a mixed number $\frac{x^2+1}{x-2}$.

Solution: Dividing x^2+1 by $x-2$, the quotient is $x+2$ with remainder $\frac{5}{x-2}$. Hence, $\frac{x^2+1}{x-2}=x+2+\frac{5}{x-2}$.

241. A mixed number can be changed to an equivalent improper fraction by multiplying the integer by the denom-

inator and taking the algebraic sum of the product and the numerator of the fraction.

Example. Reduce to improper fraction $x-1-\frac{2}{x+1}$.

Solution: $(x-1)(x+1)=x^2-1$. Add -2 and the result is x^2-3.

Hence $x-1-\frac{2}{x+1}=\frac{x^2-3}{x+1}$.

In operating with fractions the following principles are assumed as true:

242. Principle I. *Multiplying both numerator and denominator of a fraction by the same quantity changes the form of the fraction without altering its value.*

243. Principle II. *Dividing both numerator and denominator of a fraction by the same quantity changes the form of the fraction without altering its value.*

It is evident that the operations performed in Principles I and II are nothing more than multiplying or dividing the fraction by unity, and hence they do not change the value of the fraction.

REDUCTION OF FRACTIONS TO LOWEST TERMS

244. If the numerator and the denominator of a fraction have no common integral factor other than unity, then the fraction is said to be in its *lowest terms*.

245. It follows, then, that to reduce a fraction to its lowest terms it is only necessary to divide both numerator and denominator by their common factors, or to divide both terms by their h. c. f., which by Principle II above, leaves an equivalent fraction of different form.

Example 1. Reduce to lowest terms $\frac{6a^4b}{8a^3b^3}$.

Solution: Divide both numerator and denominator by $2a^3b$, their h. c. f., which gives $\frac{3a}{4b^2}$, the same fraction in lowest terms.

Example 2. Reduce to lowest terms $\frac{6a^3x-2abx}{8a^4x^2-4abx}$. Divide both numerator and denominator by $2ax$, their h. c. f., which gives:

$$\frac{6a^3x-2abx}{8a^4x^2-4abx}=\frac{3a^2-b}{4a^3x-2b}.$$

Exercise 85

Supply the missing numerators:

1. $\frac{6}{a+1}=\frac{\quad}{(a+1)(a-2)}$
2. $\frac{1}{x}=\frac{\quad}{ax}$
3. $\frac{1}{2a}=\frac{\quad}{-2a}$
4. $\frac{3}{x+3}=\frac{\quad}{x^2+3x}$
5. $\frac{1}{x-1}=\frac{\quad}{x^2-1}$
6. $\frac{a+4}{a+3}=\frac{\quad}{(a+3)\ (a-4)}$
7. $\frac{a+4}{a+3}=\frac{\quad}{a^2-a-12}$
8. $\frac{a-7}{a-4}=\frac{\quad}{a^2-a-12}$
9. $\frac{x-8}{x-4}=\frac{\quad}{4x-x^2}$
10. $\frac{5x+2}{-2x+3}=\frac{\quad}{4x^2-9}$

Reduce to lowest terms:

11. $\frac{4}{6}$
12. $\frac{a^2}{a(a+b)}$
13. $\frac{2(x-1)}{(x-1)^2(3x+1)}$
14. $\frac{(a-1)^2}{(a-1)^3(6a^2-7)}$
15. $\frac{3x-1}{9x^2-1}$
16. $\frac{a^2-b^2}{(a+b)^2}$
17. $\frac{a+b}{ax+bx+ay+by}$
18. $\frac{a^2-10a+21}{a^2+a-12}$

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

246. Common Denominator. If two or more fractions have the same denominator they are said to have a *common denominator*. Thus:

$\frac{1}{x^2-1}, \frac{x-2}{x^2-1}, \frac{2x-3}{x^2-1}$, all have the common denominator x^2-1.

Several fractions can be reduced to a set of equivalent fractions, with a common denominator.

Example. Reduce to equivalent fractions with common denominator $\frac{3}{4}, \frac{2}{3}, \frac{1}{5}$. The denominator common to these reduced fractions must be a multiple of the denominators, viz., 60.

Then $\frac{3}{4}=\frac{45}{60}$ by multiplying both numerator and denominator by 15.

Then $\frac{2}{3}=\frac{40}{60}$ by multiplying both numerator and denominator by 20.

Then $\frac{1}{5}=\frac{12}{60}$ by multiplying both numerator and denominator by 12.

Hence the set of fractions equivalent to $\frac{3}{4}, \frac{2}{3}, \frac{1}{5}$ is $\frac{45}{60}, \frac{40}{60}, \frac{12}{60}$.

247. It is generally more convenient to change any set of fractions to a set of equivalent fractions with *lowest common denominator*.

248. The lowest common denominator of a set of fractions is the *lowest common multiple* of their denominators.

The method of changing a set of fractions to equivalent fractions of lowest common denominator is as follows:

249. Rule. *Find l. c. m. of the denominators, multiply each numerator by the quotient obtained by dividing its denominator into the l. c. m. of the denominators; the resulting products placed as numerators over the l. c. m. will be the set of equivalent fractions with lowest common denominator.*

Remark. If a whole number is included in the set of fractions, write it as a fraction with denominator 1, and proceed as above.

Exercise 86

Reduce each of the following to a common denominator:

1. $\frac{6}{8}, \frac{7}{12}$
2. $\frac{1}{x}, \frac{1}{y}$
3. $\frac{2}{x^2}, \frac{1}{-x}$
4. $\frac{2}{x^2}, \frac{1}{x}$
5. $\frac{2}{a+1}, \frac{1}{a-1}$
6. $\frac{2}{x}, \frac{1}{y}, \frac{3}{z}$
7. $\frac{1}{ax}, \frac{-6}{bx}$
8. $\frac{3}{a-4}, \frac{4a}{(a-1)(a-4)}$
9. $\frac{1}{x}, \frac{3}{-2x}$
10. $\frac{2}{x-y}, \frac{-4x}{(x-y)(a-b)}$

ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS

250. Algebraic fractions can be added or subtracted exactly as the numerical fractions of Chapter I. The fractions, if not of common denominator, must be reduced to equivalent fractions of common denominator. It is, however, often more convenient to reduce the fractions to their lowest common denominator. The method for addition and subtraction may then be stated as follows:

251. Rule. *Reduce the fractions to equivalent fractions with the lowest common denominator; the sum or difference of the resulting numerators, placed over the common denominator, and this fraction reduced to lowest terms, is the fraction desired.*

1. Add $\frac{2}{x+y}, \frac{1}{x-y}, \frac{1}{x^2-y^2}$

Of these fractions the lowest common denominator is x^2-y^2.

Then $\frac{2}{x+y}=\frac{2x-2y}{x^2-y^2}$ $\qquad \frac{1}{x-y}=\frac{x+y}{x^2-y^2}$ $\qquad \frac{1}{x^2-y^2}=\frac{1}{x^2-y^2}$

The sum of these fractions is

$$\frac{2x-2y+x+y+1}{x^2-y^2}=\frac{3x-y+1}{x^2-y^2}$$

Hence $\frac{2}{x+1}+\frac{1}{x-y}+\frac{1}{x^2-y^2}=\frac{3x-y+1}{x^2-y^2}$

2. From $\frac{1}{a+b}$ take $\frac{1}{a-b}$

The lowest common denominator is a^2-b^2.

Then $\frac{1}{a+b}=\frac{a-b}{a^2-b^2}$

$\frac{1}{a-b}=\frac{a+b}{a^2-b^2}$

Then the difference between these fractions is

$$\frac{1}{a+b}-\frac{1}{a-b}=\frac{(a-b)-(a+b)}{a^2-b^2}=\frac{-2b}{a^2-b^2}$$

Hence $\frac{1}{a+b}-\frac{1}{a-b}=\frac{-2b}{a^2-b^2}$

Exercise 87

Perform the operations indicated below.

1. $\frac{2}{x+1}+\frac{1}{x+2}$
2. $\frac{1}{ax}+\frac{1}{ab}$
3. $\frac{1}{x-1}+\frac{1}{x+1}$
4. $\frac{2}{x+1}+\frac{1}{x+1}$
5. $\frac{1}{x+2}+\frac{1}{x+5}$
6. $\frac{1}{ax}-\frac{1}{ab}$
7. $\frac{1}{x-1}-\frac{1}{x+2}$
8. $\frac{x-1}{x^2-4}+\frac{x+2}{x-2}$
9. $\frac{-2}{a}+\frac{1}{-a}$
10. $1+\frac{2}{a-1}$

11. $2+\frac{3}{a-1}+\frac{1}{a^2-1}$

17. $\frac{x^2+1}{x^2-1}+\frac{4x}{x^2+1}$

12. $\frac{1}{2x+1}+\frac{1}{2x-1}$

18. $\frac{3x+1}{3x-1}-\frac{4x}{1-9x^2}+13$

13. $\frac{3}{2x+1}+\frac{5}{2x-1}-\frac{2}{4x^2-1}$

19. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$

14. $\frac{16}{a-2}-\frac{13}{(a+1)(a-2)}$

20. $\frac{x+y}{x-y}-\frac{x-y}{x+y}$

15. $\frac{a}{a-b}-\frac{a}{b}+\frac{1}{b}$

21. $\frac{a^2-a-12}{a^2+12a+36}-\frac{a^2-36}{a^2+2a-24}$

16. $\frac{c}{a}+\frac{c}{b}+\frac{c}{d}$

22. $\frac{x^2+1}{x^2+5x}+\frac{x+2}{2x+10}$

23. $\frac{3(x+y)}{2(x-y)}-\frac{6(x-y)}{5(x+y)}-\frac{6xy}{10(x^2-y^2)}$

24. $\frac{x^2+7x-8}{2x^2+11x-6}+\frac{5-8x-4x^2}{2x^2+5x-3}$

25. $\frac{1}{3}-\frac{2x}{y-x}+\frac{6xy}{x^2-y^2}+\frac{-y}{x+y}$

26. $\frac{x^2-3xy}{x^3+y^3}+\frac{3}{x+y}-\frac{x+y}{x^2-xy+y^2}$

27. $\frac{2a}{a^3-8b^3}-\frac{3(b+1)}{a^2-4b^2}+\frac{b-a}{a^2-2ab+4b^2}$

28. $\frac{2}{x+4}-\frac{x-2}{x^2-4x+16}-\frac{2x^2-16x-30}{x^3+64}$

29. $\frac{3}{x^2-11x+30}-\frac{11}{x^2-25}+\frac{1}{x^3-125}$

30. $\frac{r^2+4}{r^2-4}+1$

31. $10+\dfrac{5}{x^2-49}+\dfrac{3}{7x^2-343}$

32. $\dfrac{a}{1+rt}+\dfrac{c}{1+RT}$

33. $\dfrac{x+3}{x^2+5x+6}-\dfrac{(x+4)^2}{(x+2)^2}-\dfrac{x}{2x^2(x^2+4x+4)}$

MULTIPLICATION OF ALGEBRAIC FRACTIONS

252. In Chapter I it was found that to determine the product of two or more numerical fractions it was only necessary to multiply together the numerators for the numerator of the product, and the denominators for the denominator of the product. The resulting fraction reduced to lowest terms is the product.

Precisely the same method is followed in the multiplication of algebraic fractions.

Example. What is the product of

$$\frac{a}{a+b},\ \frac{c}{b},\ \frac{a}{a-b}?$$

The product of the numerators is a^2c.

The product of the denominators is $b(a+b)(a-b)=b(a^2-b^2)$.

Hence the product desired is

$$\frac{a^2c}{b(a^2-b^2)}$$

253. Mixed Numbers. In case one or more of the multipliers is a mixed number, reduce it to an improper fraction before multiplying; if one or more of the multipliers are integers write each of them in fraction form with denominator 1, then proceed as before.

254. Cancellation. The work of multiplication may often *be shortened* by striking out common factors from numerator

and denominator. This process of striking out common factors is known as *cancellation.*

Example. What is the product of

$$\frac{a^2}{(b+c)^2}\ \frac{b(b+c)}{a}\ \frac{c}{2(b-c)b}$$

Here $b+c$ may be cancelled from the numerator of the second fraction and from the denominator of the first, a from numerator of the first and denominator of the second, and b from the numerator of the second and denominator of the third. The result then is (compare § 21)

$$\frac{ac}{2(b+c)(b-c)}=\frac{ac}{2(b^2-c^2)}$$

Exercise 88

Perform the operations indicated.

1. $\dfrac{2x^3y}{3z^3}\cdot\dfrac{9z^2}{4x^3y}$

2. $\dfrac{10x}{15y}\cdot\dfrac{18ax}{7b}\cdot\dfrac{8b^2y}{12x}$

3. $\dfrac{3a^2b^3}{4b^2}\cdot\dfrac{32bc}{9a}$

4. $\dfrac{5}{x^3+8}\cdot(x-1)$

5. $\dfrac{6z}{x}\cdot\dfrac{(3x)^2}{(z)}\cdot\dfrac{(1)^2}{(2)}\cdot\dfrac{2x}{3z}$

6. $\dfrac{(2ab)^2}{(15xy)}\cdot\dfrac{(-x)^3}{(3a)}\cdot\dfrac{9y^2}{2}$

7. $(x-y)\cdot\dfrac{2y+xy-x^2}{y-x}$

8. $(a^3-b^3)\cdot\dfrac{3}{b-a}$

9. $\dfrac{2a-3}{a^2+5a+6}\cdot\dfrac{a^2+8a+15}{a+1}$

10. $\dfrac{2a-3}{a^2-5a+6}\cdot(a^2-8a+15)$

11. $\dfrac{x^2+10x+21}{x^2-25}\cdot\dfrac{x+5}{x^2+4x-21}$

12. $\dfrac{r}{9-t^2}\cdot\dfrac{t^2+5x+6}{27rt}$

13. $\left(\dfrac{3x}{4}-2\right)^2\cdot\dfrac{8a^2}{9x^2-64}\cdot\dfrac{3x+8}{3ay}$

14. $\dfrac{5-20x}{1+8x^3}(15-9x^2)\dfrac{1+2x}{1-4x}$

15. $\dfrac{a+b}{a^6-b^6}\cdot\dfrac{a^3+b^3}{a^2-b^2}$

16. $\left(\dfrac{8x-12}{2x-3}+2x\right)\left(2x+3+\dfrac{16}{2+2x}\right)$

17. $(27x^3+1)\cdot\dfrac{a+1}{9a^2-3a+1}$

18. $\dfrac{r-3t}{r^2-6rt+9t^2}\left(4r-\dfrac{t^2}{r}\right)\dfrac{4r^2+t}{16r^4-t^4}$

19. $\left(1-\dfrac{2}{x}-\dfrac{3}{x^2}\right)\cdot\dfrac{19}{(x^2-14x+33)}$

DIVISION OF FRACTIONS

255. The principle involved is exactly the same as in Chapter I, namely:

Rule. *To divide an expression, either integral or fractional, by a fraction, invert the divisor and multiply.*

Divide $\dfrac{a^3+b^3}{a^2-9b^2}$ by $\dfrac{a+b}{a+3b}$

$$\frac{a^3+b^3}{a^2-9b^2}\div\frac{a+b}{a+3b}=\frac{a^3+b^3}{a^2-9b^2}\cdot\frac{a+3b}{a+b}=$$

$$\frac{(a+b)(a^2-ab-b^2)}{(a-3b)(a+3b)}\cdot\frac{a+3b}{a+b}=\frac{a^2-ab+b^2}{a-3b}$$

Exercise 89

Simplify the following:

1. $\dfrac{x^2+2x+1}{x^2+2x} \div \dfrac{3x^3+3x^2}{x^2-6x-16}$

2. $\dfrac{3a^2b}{8c^2} \div \dfrac{(6ab^2)^2}{-2bc^2}$

3. $\dfrac{12a-4}{15c^2x} \div \dfrac{3a-1}{5c}$

4. $\dfrac{x^2-x-2}{x^2+4x-12} \div \dfrac{x^2+2x}{x^2-4}$

5. $\dfrac{2x^2-8}{x^2+3x} \div \dfrac{x^2+4x+4}{3x^3-3x^2-36x}$

6. $\dfrac{4r^2t-4rt^2-3t^3}{12rt^2} \div \left(\dfrac{9t^2}{4r}-r\right)$

7. $\dfrac{4x^2-y^2}{x^2y} \div \dfrac{2xy-y^2}{x^4-y^4} \div \dfrac{y^2(x^2+y^2)}{x^2y^2}$

8. $\dfrac{a^2-3a+18}{a^4-13a^3+42a^2} \div \dfrac{4a^2+32a+60}{6a^3-66a^2+168a}$

When successive division signs occur, the operations must be performed in their order from left to right, that is, the second division is performed on the result of the first, thus:

$$\frac{a}{b} \div \frac{c}{d} \div \frac{x}{y} = \frac{a}{b} \times \frac{d}{c} \div \frac{x}{y} = \frac{ad}{bc} \div \frac{x}{y} = \frac{ad}{bc} \cdot \frac{y}{x} = \frac{ady}{bcx}$$

Below is an incorrect solution:

$$\frac{a}{b} \div \frac{c}{d} \div \frac{x}{y} = \frac{a}{b} \div \frac{c}{d} \cdot \frac{y}{x} = \frac{a}{b} \div \frac{cy}{dx} = \frac{a}{b} \cdot \frac{dx}{cy} = \frac{adx}{bcy}$$

9. $\dfrac{(6z)^2}{(x)} \div \dfrac{(-3z)}{(2x)} \div \dfrac{(3x)^4}{(2z)}$

10. $\dfrac{x^3-y^3}{x^2+xy} \div \left(\dfrac{x}{y}-\dfrac{y}{x}\right) \div \left[xy\left(x+y+\dfrac{y^2}{x}\right)\right]$

11. $\dfrac{7x^3-7x^2}{x^2-8x-9} \div \dfrac{5x^5+40x^4-45x^3}{x^2+2x-99}$

12. $\dfrac{x^2y^2(z^2+z-20)}{z^2-12z+32} \cdot \dfrac{z^2+z-72}{z^2+11z+28} \div \dfrac{x^3y^3(z^2+8z-9)}{z^2+8z+7}$

COMPLEX FRACTIONS

256. Definitions. Sometimes fractions occur whose numerators or denominators or both are fractional in form. Such fractions are called *complex fractions.*

Examples.

$$\frac{\frac{1}{2}+\frac{1}{3}}{3}, \quad \frac{\frac{a}{a+b}-\frac{c}{a}}{\frac{a-b}{2}}$$

Complex fractions can be simplified by the following rule:

257. Rule. *Reduce all the fractions of the numerator and denominator to the same denominator, and multiply both terms of the fraction by this common denominator.*

Example. Simplify

$$\frac{\frac{a}{1+a}+\frac{1-a}{a}}{\frac{a}{1+a}-\frac{1-a}{a}} = \frac{\frac{a^2}{a(1+a)}+\frac{(1-a)(1+a)}{a(1+a)}}{\frac{a^2}{a(1+a)}-\frac{(1-a)(1+a)}{a(1+a)}}$$

$$= \frac{a^2+(1-a)(1+a)}{a^2-(1-a)(1+a)} = \frac{a^2+1-a^2}{a^2-1+a^2} = \frac{1}{2a^2-1}$$

Exercise 90

1. When is a complex fraction in its simplest form?

Simplify the following:

2. $\dfrac{1+\frac{1}{a}}{1-\frac{1}{a}}$

3. $\dfrac{\frac{1}{x}}{1-\frac{1}{x^2}}$

4. $\dfrac{6a-+1}{\frac{(a+3)^2}{4a^2}}$

5. $\dfrac{2+\frac{c+d}{4}}{2-\frac{c-d}{4}}$

6. $\dfrac{1-\dfrac{b^2+c^2-a^2}{2bc}}{\dfrac{3}{2bc}}$

7. $\dfrac{\dfrac{r}{r+3}-\dfrac{r}{r-2}}{\dfrac{r(r+2)}{r+3}-\dfrac{r(r+2)}{r-2}}$

8. $\dfrac{\dfrac{1}{x+3}-\dfrac{1}{x-5}}{\dfrac{3x^2}{x^3-2x^2-15x}}$

9. $1+\dfrac{1}{1+\dfrac{1}{x}}$

10. $\dfrac{1-\dfrac{x^2-1}{x^2-6x+5}}{1-\dfrac{2x+3}{x-5}}$

11. $\dfrac{4+\dfrac{2a}{b}}{\dfrac{(a-2b)^2}{4b}+2a}$

12. $\dfrac{\left(\dfrac{1}{x}-x\right)^2+2}{6\left(\dfrac{1}{x^2}+x^2\right)}$

13. $\dfrac{\dfrac{a}{x^3+3ax^2+3a^2+a^3}}{\dfrac{16a^3}{x^2-a^2}}$

14. $\dfrac{x+\dfrac{x^2+y^2}{y}}{\dfrac{x^3}{x^3-y^3}-1}$

15. $\dfrac{\dfrac{x^2+3xy-7}{2x^2}}{\dfrac{3xy-7x^2+4}{5}}$

16. $\dfrac{2+\dfrac{1}{a-2}+a}{a+\dfrac{1}{a-2}-2}$

RATIO AND PROPORTION

258. Definitions. The *ratio* of one number to another is the quotient of the first divided by the second. This quotient even if it is easily found is written as a fraction.

For example, the ratio of a to b is $\dfrac{a}{b}$. The ratio of 3 to 5 is $\frac{3}{5}$. The

ratio of 8 to 4 is $\frac{8}{4}$. In this book all ratios are regarded as fractions and all fractions may be regarded as ratios.

259. A ratio is always an *abstract* number, that is, it never has a name such as feet, dollars, hours, or pounds. A ratio may be made up of *concrete* numbers, in fact, its terms usually are concrete, but the ratio regarded as a whole is abstract. For example, the ratio of 8 feet to 4 feet is $\frac{8}{4}$ or 2, *but not 2 feet.*

260. The numerator is called the *antecedent* of the ratio; the denominator is the *consequent.*

261. An equation, each of whose members is a ratio, is called a *proportion.*

Thus $\frac{a}{b}=\frac{c}{d}$ is a proportion.

It is read: The ratio of a to b equals the ratio of c to d, or, simply, a is to b equals c is to d.

The four numbers, a, b, c, d, are said to be *in proportion.*

262. Of the four quantities in a proportion the first and last (as a and d above) are called the *extremes*; the second and third (as b and c above) are called the *means.*

263. Theorems on Proportions. If in the proportion

$$1. \qquad \frac{a}{b}=\frac{c}{d}$$

both members are multiplied by bd we get

$$\frac{abd}{b}=\frac{cbd}{d}.$$

Cancelling like factors in the numerators and denominators.

$$2. \qquad ad=cb.$$

When we recall the definitions of means and extremes given above, this result can be stated in words as follows:

264. Theorem. *In any proportion the product of the means equals the product of the extremes.*

A statement of this sort, which has been deduced from given facts, is called a *theorem.*

If 1 be added to both members of the equation $\frac{a}{b}=\frac{c}{d}$ thus, $\frac{a}{b}+1=\frac{c}{d}+1$, and both members be reduced to their common denominators, we get

$$3. \quad \frac{a+b}{b}=\frac{c+d}{d}$$

This is called *addition,* or very commonly *composition.* Let the student state the corresponding theorem in words.

Exercise 91

Solve the following proportions for x:

1. $\frac{2}{3}=\frac{6}{x}$

Solution: $\frac{2}{3}=\frac{6}{x}$ Check: $\frac{2}{3}=\frac{6}{9}$

By §264, $2x=18$

Dividing, $x=9$

2. $\frac{3}{5}=\frac{9}{x}$

3. $\frac{4}{8}=\frac{16}{x}$

4. $\frac{4}{x}=\frac{8}{16}$

5. $\frac{6}{11}=\frac{x}{20}$

6. $\frac{\frac{1}{x}}{2}=\frac{2}{31}$

7. By subtracting 1 from both members of Equation 1, § 263, prove $\frac{a-b}{b}=\frac{c-d}{d}$. This is called *subtraction,* or very commonly *division.*

State the theorem suited to the result found in this exercise.

8. Divide Equation 3, § 264, by $\frac{a-b}{b}=\frac{c-d}{d}$ and find $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. This is called *addition and subtraction*, or *composition and division.*

9. $\frac{a}{b}=\frac{c}{x}$. Solve for x.

10. $\frac{a}{b}=\frac{b}{x}$. Solve for x.

11. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a}{b}=\frac{kc}{kd}$ where k is any number.

265. Problems on the Seesaw. Suppose two boys place a plank across a support, as shown in the figure, and seesaw. The point of support, F, is called the *fulcrum.*

Every boy knows that, for balance, the larger person must sit nearer the fulcrum. That is, if w_1 is larger than w_2 then d_1 must be smaller than d_2. The exact relation between these four quantities may be determined by weighing the quantities w_1 and w_2, placing them on the seesaw so that they balance, and carefully measuring the distances d_1 and d_2. Physicists who have performed this experiment many times, tell us that, for balance, the following equation holds:

$$1.\quad \frac{d_1}{d_2}=\frac{w_2}{w_1}$$

Notice also that if w_1 is larger than w_2 in Equation 1, then d_2 must be larger than d_1, so that the two larger numbers may be in the denominators. This agrees with our experience.

It is the practice among scientific men to express general laws by literal equations, such as 1. Since d_1, d_2, w_1, and w_2 may have any values whatever (subject to the restriction that they satisfy the proportion of Equation 1), this equation holds in all cases and is therefore a complete expression of the "general" law.

If any three of the quantities d_1, d_2, w_1, and w_2 are known, the fourth can always be found by solving Equation 1.

Exercise 92

Find d_1 when:

1. $d_2=6$ ft., $w_1=40$ lb., $w_2=56$ lb.
2. $d_2=7$ ft., $w_1=112$ lb., $w_2=84$ lb.
3. $d_2=4.6$ ft., $w_1=96.4$ lb., $w_2=78.3$ lb.

Find d_2 when:

4. $d_1=8.1$ ft., $w_1=66$ lb., $w_2=49$ lb.
5. $d_1=5.9$ ft., $w_1=38$ lb., $w_2=57$ lb.

Find w_1 when:

6. $d_1=6$ ft., $w_2=55$ lb., $d_2=12$ ft.

Find w_2 when:

7. $d_1=5$ ft., $d_1+d_2=11$ ft., $w_1=114$ lb.
8. Solve equation $\frac{d_1}{d_2}=\frac{w_2}{w_1}$ for d_1 in terms of d_2, w_2, and w_1.
9. Solve $\frac{d_1}{d_2}=\frac{w_2}{w_1}$ for d_2, in terms of the other three quantities.

Solve also for w_2. Solve for w_1.

10. Two boys weighing 55 and 65 lb. respectively sit on ends of a seesaw 14 ft. long. Find d_1 and d_2 so that there is a balance

11. A and B weigh together 210 lb. If they balance when A is 4 ft. from the fulcrum and B is 7 ft., how much does each weigh?

12. Some boys use their 16 ft. seesaw board in an attempt to pry up a large stone. They find a convenient crack in the stone, place their fulcrum 3 ft. from the end, and two of them weighing together 250 lb. sit on the other end. (See figure.) Find the pressure they exert on the stone.

13. The boys of problem 12, finding that they can not move the stone, put their fulcrum 2 ft. from the stone. Find the pressure they exert on it.

14. Solve problem 12 if the fulcrum is put 1 ft. from the stone; 6 in.; 3 in.

15. If two boys, A and B, whose weights are different, are balanced on a seesaw, can two more boys, one weighing exactly as much as A and the other exactly as much as B sit with them without destroying the balance? (See problem 11, Ex. 91.)

16. If a building 75 ft. high casts a shadow 125 ft. long, how high is a tree which has a shadow 105 ft. long?

17. A string 24 in. long is divided in the ratio of 3 to 5. What is the length of each part?

18. If 26 men build a wall in 21 days, how many men could have built it in 14 days?

19. A tower 100 ft. high and 25 ft. in diameter is represented by a model 8 in. high. How many inches is the diameter of the model?

20. A bankrupt owes A $300.00, B $350.00, and D $450.00. Divide $600.00 among A, B, and C in proportion to their claims.

21. If 56 bushels of corn costs \$64.00 what will 70 bushels cost?

22. Where is the point of support if two boys, one weighing 125 lb. and the other 95 lb. exactly balance on the ends of a 14 ft. pole? (Disregard weight of the pole.)

23. A rectangular field has a perimeter of 600 rods. Its length is to its width as 3 is to 2. Find its dimensions.

24. If a man receives \$30.00 for plowing 25 acres what ought he to receive for plowing $36\frac{1}{2}$ acres?

25. A field produces timothy and clover in the ratio of 5 to 7. It produces $8\frac{1}{2}$ tons of timothy. How much clover does it produce?

26. In a proportion the means are 12 and 19. The first extreme is 9. What is the other?

27. If $\frac{a}{b}=\frac{c}{d}$ prove that $\frac{a+2b}{b}=\frac{c+2d}{d}$, also $\frac{a-3b}{b}=\frac{c-3d}{d}$

28. Divide 15 into two parts in the ratio of 2 to 3.

29. If 5 horses are worth 70 sheep, how many sheep are worth 9 horses?

30. When $\frac{a}{b}=\frac{c}{d}$, show that $\frac{3a-4b}{3b}=\frac{3c-4d}{3d}$.

266. Summary of Algebraic Fractions. This chapter is a generalization of Chapter I. A thoughtful student should be able to see that algebra is generalized arithmetic. A repetition of the rules for addition, subtraction, multiplication, and division of fractions is unnecessary since they are essentially the same as in Chapter I. The student will note that the long drill on factoring was necessary in order that he might be able to simplify algebraic fractions.

There are two new subjects introduced, viz.: proportion and problems from physics. The first is a part of every course

in algebra. Its most important theorems may be summarized as follows:

1. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Addition)

2. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Subtraction)

3. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Addition and Subtraction)

The problems on the seesaw ought to be carefully solved, because they give a simple application of the results achieved in the previous study of algebra. *To be able to use algebra is the only sure proof that one knows the subject.*

Exercise 93 — Review

Simplify the following fractions:

1. $(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)$

2. $(a-b)\left(\frac{1}{b}-\frac{1}{a}\right)$

3. $\left(\frac{2a}{3b}-a\right)\left(a-\frac{ab}{2}\right)$

4. $\left(\frac{x}{y}-1\right)\left(\frac{y}{x}+1\right)$

5. $\left(\frac{a}{b}-\frac{b}{a}\right)\div\left(\frac{b}{a}-\frac{a}{b}\right)$

6. $\left(a^2-\frac{1}{9b^2}\right)\div\left(a-\frac{1}{3b}\right)$

7. $\left(\frac{a^2}{b^2}+\frac{a}{b}+1\right)\left(\frac{a^2}{b^2}-\frac{a}{b}+1\right)$

8. $2\left[\frac{1}{a-3}-\frac{1}{a-2}\right]-\frac{2}{a^2-5a+6}$

9. $\left(\frac{a}{2c}+\frac{b}{3c}\right)^2-\frac{ab}{3c^2}$

10. $\left(\frac{x^3y^3}{z^3}+\frac{x^2y^2}{z^2}+\frac{xy}{z}+1\right)\left(\frac{xy}{z}-1\right)$

11. $\left(x^3-\frac{1}{x^3}\right)\div\left(x-\frac{1}{x}\right)$

12. $\frac{5}{x+7}-\frac{3}{x-5}-\frac{2}{x}$

13. $\left(\frac{3x+3y}{3x-4y}-\frac{3x-3y}{3x+4y}\right)\div\frac{4x^2-9y^2}{9x^2-16y^2}$

14. $\left(\frac{a+b}{a-b}\right)^2-\left(\frac{a-b}{a+b}\right)^2$

15. $\left(a^3-\frac{1}{a^3}\right)\div\left(a^2+1+\frac{1}{a^2}\right)$

16. $\dfrac{a}{b+\dfrac{1}{c+\dfrac{1}{a}}}$

17. $\frac{2}{x-1}-\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^3}$

18. $\frac{2}{5x^2-10x+5}+\frac{11}{25x-25}$

19. $\dfrac{1}{1-\dfrac{1}{a}}+\dfrac{1}{1+\dfrac{1}{a}}$

20. $\frac{5x^2+4x-1}{5x^2+19x-4}$

21. $\frac{x-3}{x+1}\cdot\frac{x^2+2x+1}{x^3-27}$

22. $\frac{x^3-1}{x^2-a^2}\div\frac{x^2+x+1}{x-a}$

23. $\frac{1}{x-y}-\frac{3xy}{x^3-y^3}$

24. $\dfrac{x-y-\dfrac{x^2+y^2}{x-y}}{y-x+\dfrac{x^2}{x+y}}$

25. $a^2-ab+b^2-\frac{2b^3}{a+b}$

26. $\dfrac{1}{x^2-\dfrac{x^3-1}{x+1}}$

27. $\frac{x-1}{x-2}-\frac{x-3}{x-1}$

28. $\dfrac{\dfrac{x}{3}-\dfrac{y}{4}}{5}-\dfrac{\dfrac{x}{5}-\dfrac{y}{6}}{3}+\dfrac{8x-y}{15}$

29. $\dfrac{1-\left(\dfrac{1-x}{1+x}\right)^2}{1+\left(\dfrac{1-x}{1+x}\right)^2}$

30. $\left(\dfrac{a-1}{a+2}\right)^2\left(\dfrac{a^2-4}{a^2-1}\right)$

31. $\dfrac{1}{a-1}-\dfrac{a^2+2a}{a^3-1}$

32. $\dfrac{1}{(x-1)^2}\cdot\dfrac{x^2-6x+5}{6(x+1)}\cdot\dfrac{5(x^2-1)}{x}$

33. $\dfrac{a^3-1}{a^2+a+1}$

34. $\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$

35. $\left(\dfrac{a}{1+a}+\dfrac{1-a}{a}\right)\div\left(\dfrac{a}{1+a}-\dfrac{1-a}{a}\right)$

36. $\left(1-\dfrac{2xy}{x^2+y^2}\right)-\left(1-\dfrac{y}{x}\right)$

37. $\dfrac{(x^2-5x)^2}{x^2-10x+25}$

38. $\dfrac{a^2-(b-c)^2}{c^2-(b-a)^2}\div\dfrac{(b-c)^2-a^2}{(a-b)^2-c^2}$

39. $\left(\dfrac{1}{9}-\dfrac{2x}{15}+\dfrac{x^2}{25}-\dfrac{x^4}{49}\right)\div\left(\dfrac{1}{3}-\dfrac{x}{5}+\dfrac{x^2}{7}\right)$

40. $\dfrac{x^2+7x+12}{x^2+2x-15}\cdot\dfrac{x+5}{x+4}$

CHAPTER XIII

EQUATIONS INVOLVING FRACTIONS

TRANSPOSITION OF TERMS

267. The student has been taught so far that to simplify and solve an equation he may add or subtract the same quantity in both members, and multiply or divide both members by the same quantity. A somewhat simple form of the process for adding and subtracting will now be illustrated.

Example. Solve $3x-16=2x-\frac{7}{2}$

Subtract $\qquad 2x=2x$

$$3x-2x-16=0-\frac{7}{2}$$

Notice that subtracting $2x$ from both members is exactly equivalent to transposing the term $2x$ from the right member to the left and changing its sign from plus to minus.

Thus, $\qquad 3x-16=2x-\frac{7}{2}$

Transposing, $\qquad 3x-2x-16=\frac{-7}{2}$

Transposing again, $\qquad 3x-2x=16-\frac{7}{2}$

Therefore $\qquad x=\frac{25}{2}$

Since subtraction is equivalent to addition of the same number with its sign changed, we have the following:

268. Principle. *Any term of an equation may be transposed to the other side of the equality marks provided the sign of the term be changed.*

Illustration. The following example solved by both methods will make this new method clearer.

Example. Solve for x, the equation $6x+9=5(x-3)+30$.

Old Method

$$\begin{array}{lrcl}
 & 6x+9 & = & 5(x-3)+30 \\
 & 6x+9 & = & 5x-15+30 \\
 & 6x+9 & = & 5x+15 \\
\text{Subtract} & 9 & = & 9 \\ \hline
 & 6x & = & 5x+6 \\
\text{Subtract} & 5x & = & 5x \\ \hline
 & x & = & 6
\end{array}$$

Check: $6\cdot 6+9=5(6-3)+30$

$45=5\cdot 3+30$

$45=45$

New Method

$$\begin{array}{lrcl}
 & 6x+9 & = & 5(x-3)+30 \\
 & 6x+9 & = & 5x-15+30 \\
\text{Transposing} & 6x-5x & = & +30-15-9 \\
 & x & = & 6
\end{array}$$

Check: $6\cdot 6-9=5(6-3)+30$

$45=5\cdot 3+30$

$45=45$

269. An equation may be freed from fractional coefficients by multiplying both members of it by the l. c. m. of the denominators.

Example. Solve $\dfrac{4x+1}{6}-\dfrac{1}{5}(x-3)=\dfrac{4}{15}$

Multiplying by 30, $5(4x+1)-6(x-3)=8$

$20x+5-6x+18=8$

Transposing, $20x-6x=8-5-18$

$14x=-15$

$x=-\dfrac{15}{14}$

Check: $$\frac{4\left(-\frac{15}{14}\right)+1}{6}-\frac{1}{5}\left(-\frac{15}{14}-3\right)=\frac{4}{15}$$

$$\frac{-\frac{30}{7}+1}{6}+\frac{3}{14}+\frac{3}{5}=\frac{4}{15}$$

$$-\frac{23}{42}+\frac{3}{14}+\frac{3}{5}=\frac{4}{15};\quad -\frac{23}{42}+\frac{9}{42}+\frac{3}{5}=\frac{4}{15};\quad -\frac{14}{42}+\frac{3}{5}=\frac{4}{15};$$

$$-\frac{1}{3}+\frac{3}{5}=\frac{4}{15};\quad \frac{-5+9}{15}=\frac{4}{15};\quad \frac{4}{15}=\frac{4}{15}$$

Exercise 94

In the following equations, (1) clear of fractions by multiplying both members by the l. c. m. of the denominators;

(2) solve by transposing and dividing.

Check always in the original equation.

1. $\frac{x}{3}-\frac{x}{4}=\frac{6}{2}$

2. $\frac{4x}{3}+\frac{3x}{4}=2\frac{1}{3}$

3. $\frac{4x}{3}+\frac{3x}{4}=2+\frac{1}{3}$

4. $\frac{11x}{3}+\frac{3x}{4}=\frac{7}{3}$

5. $\frac{x+1}{2}+\frac{2x-1}{3}=8$

6. $\frac{3x-1}{5}-\frac{1}{2}(x+6)=\frac{1}{2}$

7. $\frac{6x+17}{7}=3x-1+\frac{2x-1}{7}$

8. $\frac{9x-6}{5}=\frac{7}{3}\left(\frac{3}{2}-x\right)+\frac{14x-8}{15}$

9. $\frac{2p}{31}-\frac{p+11}{62}=\frac{1}{31}-p+\frac{3p-6}{2}$

10. $\frac{a+13}{13}=\frac{6-3a}{65}+1$

11. $\frac{2}{x}+\frac{3}{x}=10$

12. $\frac{1}{3x}-\frac{14}{21}=\frac{16}{x}$

13. $\frac{x}{33}=2-\frac{3x}{165}$

14. $\frac{8.3h}{33000}=\frac{20}{2175.3}$

15. $\frac{x}{a}+\frac{2x}{b}=1$

Exercise 95 — Problems

1. One half of a certain number plus one third of that number equals 10. Find the number.

Let $\qquad n =$ the number

Then $\qquad \frac{n}{2} =$ one half the number.

$\qquad \frac{n}{3} =$ one third the number

and $\qquad \frac{n}{2}+\frac{n}{3} =$ one half plus one third.

But, by the problem, $10 =$ one half plus one third.

Therefore $\qquad \frac{n}{2}+\frac{n}{3}=10$

Multiplying by 6, $3n+2n=60$

$$5n=60$$

$$n=12$$

Check: $\frac{1}{2}$ of 12 and $\frac{1}{3}$ of $12=6+4=10$.

2. Two fifths of a certain number plus one fourth of it equals 13. Find the number.

3. The difference between one eighth of a number and one tenth of it is 2. Find the number.

4. The sum of one half, one third, and two fifths of a number is $24\frac{2}{3}$. Find the number.

5. One half the difference between five times a certain number and 3 equals $\frac{2}{13}$ of the difference between 4 times the number and 5. Find the number.

6. Separate 31 into two parts so that one half of one part equals the other part minus 1.

7. The sum of two consecutive odd numbers equals two thirds the sum of the next two consecutive odd numbers. Find the numbers.

8. The difference of two consecutive odd numbers equals one fourth the sum of two other consecutive odd numbers. Find the second pair.

9. The difference of two consecutive odd numbers equals the difference between the next two consecutive odd numbers. *Can you* find the numbers?

10. A number of two digits has 3 in the units' place. If the digits be reversed the number is 5 units more than doubled. Find the number.

Let x stand for the number of units in the tens' place; then the number is $10x+3$. Compare $73=70+3=10\times7+3$.

11. A number of two digits has 2 in the tens' place. If to one third this number 7 be added, twice the sum equals the original number plus 6. Find the number.

12. The units' digit of a number is 4 times the tens' digit. If 1 is added to each digit the number will equal $\frac{3}{2}$ the original number, minus 3. Find the number.

13. A number of two digits has the units' digit one third the tens' digit. If the unit's digit is increased by 6, the number is doubled. Is there any such number?

14. There are two numbers whose sum is 24. If their difference is divided by their sum the quotient is $\frac{1}{60}$ of the greater. Find the numbers.

15. A rifleman at target practice hears the bullet strike the target $4\frac{1}{2}$ seconds after the discharge. If the average velocity of the bullet is 1800 feet per second and the velocity of sound is 1024 feet per second, find the distance to the target and the time the bullet is in the air.

16. A gunner of a battleship shooting at a target 6 miles away saw his shell strike the target 31 seconds before the explosion of the shell at the target was heard. With what velocity did the sound travel?

17. A can do a piece of work in 6 days. B can do it in 5 days. In how many days can they do it working together?

Let x = the number of days required.

They can do $\frac{1}{x}$ of it in one day. But A can do $\frac{1}{6}$ of it in one day and

B can do $\frac{1}{5}$ of it in one day. Therefore both can do $\frac{1}{6}+\frac{1}{5}$ of it in one day. Hence $\frac{1}{6}+\frac{1}{5}=\frac{1}{x}$

Multiplying by $30x$, $5x+6x=30$; $11x=30$; $x=\frac{30}{11}$.

Check: $\frac{1}{6}+\frac{1}{5}=\frac{11}{30}$; $\frac{5+6}{30}=\frac{11}{30}$; $\frac{11}{30}=\frac{11}{30}$.

18. A can do a piece of work in 12 days and B in 6 days. How long will it take both together to do it?

19. A and B working together can do a piece of work in 4 days. A alone can do it in 7 days. How long will it take B alone to do it?

20. A pipe can fill a cistern in 11 hours. A second pipe can fill it in 3 hours. How long will it take them both to fill it?

21. A pipe can fill a cistern in 12 hours; a second pipe can fill it in 9 hours. How long will it take them both to fill it?

FRACTIONAL EQUATIONS WITH POLYNOMIAL DENOMINATORS

270. The same rule holds for polynomial denominators that held for equations with simple denominators, viz.:

Find the l. c. m. of all the denominators and clear of fractions by multiplying both members of the equation by the l. c. m.

271. It is in this process that we see our best reason for a check, for whenever an equation is multiplied by an expression containing the unknown there may be introduced an apparent root of the equation which will not satisfy the equation. Such a root is called an *extraneous root* of the equation. Any result we obtain is a root, solely because it will satisfy the equation, and not because we have gone through some particular process to get it.

Example. $\frac{2x+5}{x+3}=\frac{-1}{x+3}+1.$ Multiplying both members by $x+3$ we

get $2x+5=-1+x+3$. Transposing, $x=-5-1+3$; $x=-3$. On attempting to check, we find

$$\frac{-1}{0}=\frac{-1}{0}+1$$

and this may or may not be an equation.

But division by zero is meaningless, since the quotient times the divisor could never equal the dividend (see definition of divisor). Therefore this solution does not check and this equation has no solution. That the equation is impossible can be seen more clearly by collecting before clearing of fractions as follows:

$$\frac{2x+5}{x+3}=\frac{-1}{x+3}+1;\quad \frac{2x+5}{x+3}+\frac{1}{x+3}=1;\quad \frac{2x+6}{x+3}=1;\quad \frac{2(x+3)}{x+3}=1;$$

or, $2=1$.

Exercise 96

Solve and check:

1. $\dfrac{2x+6}{3+2x}=4$

2. $\dfrac{1}{x+6}+\dfrac{5}{2x+12}=-1$

3. $\dfrac{3}{1+x}+\dfrac{x^2}{1-x^2}+\dfrac{2}{1+x}=-1$

4. $\dfrac{1}{x-4}+\dfrac{2}{x+4}=\dfrac{-3}{x^2-16}$

5. $\dfrac{x-4}{x-5}+\dfrac{x-15}{x+4}=\dfrac{2x^2-10x-1}{x^2-x-20}$

6. $\dfrac{x+1}{x+2}+\dfrac{x+3}{x+1}=\dfrac{2x^2+18}{x^2+3x+2}$

7. $\dfrac{x+1}{x-1}=\dfrac{a+b}{a-b}$

8. $\dfrac{2x-1}{x+2}-\dfrac{4x-1}{2x-3}=\dfrac{-10}{(x+2)(2x-3)}$

9. $\dfrac{17}{40x-120}+\dfrac{30}{17}=\dfrac{3}{15-5x}$

LITERAL EQUATIONS

272. Literal Coefficients. Most of the problems and examples up to this point have had arithmetical numbers for their coefficients. There were certain cases, however, where the coefficients were letters, notably the formula

$$a=p+prt$$

which we solved for p, in terms of a, r, and t; for r in terms of a, p, and t; and for t in terms of a, p, and r. These solutions illustrate the value of literal equations. *The solution of a literal equation combines in one step all possible cases of certain types of problems.*

273. For example, a great advantage is gained in clearness and in time saved by having the literal equation $a=p+prt$, which covers all possible cases. The solution of this equation for p gives a formula for the principal which holds in all cases.

274. Again, the results of a number of numerical cases led us to the formula $v_2t=v_1t+n$ (see Exercise 60), which we solved for t and n in terms of the other symbols. The ability to solve literal equations and simplify literal expressions saves in many cases a tremendous amount of calculation. A student who can solve a literal equation for any one of the letters in terms of the others has made considerable progress in algebra.

Exercise 97

Solve the following equations:

1. $\dfrac{ax+6}{3+ax}=2a$

2. $\dfrac{1}{x+c}=\dfrac{5}{2(x+c)}-1$

3. $\frac{a}{x}+\frac{2a}{3x}=\frac{3}{7}$

4. $\frac{p}{a}+\frac{p}{b}=\frac{1}{ab}$. Solve for p.

5. $\frac{x}{a}+\frac{y}{b}=1$. Solve for x.

6. $\frac{x}{a}+\frac{y}{b}=1$. Solve for y.

7. $2ax-\frac{3b}{3}\left(x-\frac{3a}{4}\right)=10\left(\frac{a}{2}-\frac{2b}{5}\right)$

8. $a=p+prt$. Solve for p; for r; for t.

9. $s=\frac{n}{2}(a+1)$. Solve for n; for a; for 1.

10. $H_p=\frac{62.5\ C_FH}{33000}$ (where H_p means horsepower, C_F means cubic feet and each is regarded as a single symbol). Solve for C_F, for H.

11. $\frac{a^2x}{2b}-3b=\frac{6bx}{5a}+6a^2$

12. $v_1t=v_2t+n$. Solve for t; for n.

13. $\frac{d_1}{d_2}=\frac{w_2}{w_1}$. Solve for d_1; for d_2; for w_1; for w_2.

14. $s=\frac{a}{1-r}$. Solve for a; for r

15. $l=a+(n-1)d$. Solve for a; for n; for d.

16. $s=\frac{rl-a}{r-1}$. Solve for l; for r; for a.

Exercise 98—Problems

Hydraulic engineers use the following formula for computing the horsepower of a stream of water:

$$H_p=\frac{62.56C_FH}{33000}$$

where H_p = horsepower, C_F = number of cubic feet of water per minute which the stream flows, H = height of the dam in feet.

1. Find the horsepower of a stream flowing 5000 cubic feet per minute, the height of the dam being 21 feet.

2. Find the height of the dam that must be built to produce 10,000 H_p from a stream flow of 150,000 cubic feet per minute.

3. The French Broad River at its lowest flows 320,000 cubic feet per minute. It is desired to produce $300{,}000H_p$. Find the height of the dam.

4. If it is not practicable to build a dam more than 15 feet high on a certain stream find the stream flow that is required to produce $700H_p$.

A degree on a Fahrenheit thermometer is $\frac{5}{9}$ a degree on a Centigrade thermometer. The zero on the Centigrade is at the freezing point of water, that is, when F. = 32, C. = 0. The Centigrade thermometer is used in nearly all experiments in chemistry and physics when it is necessary to know the temperature.

5. As the temperature rises the Fahrenheit goes up 9 degrees while the Centigrade goes up 5. Starting F. at 32 and C at 0, if C. goes up 10 what will F. read?

6. If C. goes up 15 what will F. read? If C. goes up 20 what will F. read?

7. If C. goes up $5x$ degrees what will F. read?

8. In each case above you divided the C. reading by 5, multiplied the result by 9, and added 32. Express this relation in a literal formula which will solve all problems of this type. Notice that the reasoning is very similar to the formula $v_2t = v_1t + a$ of problem 10 in Exercise 60.

9. Solve $F = 32 + \frac{9}{5}C$ for C.

10. Given $F=18$. Find C.

11. Given $F=-32$. Find C.

12. Given $F=212$. Find C.

13. Given $C=100$. Find F.

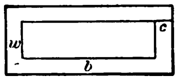
14. Given $C=-40$. Find F.

15. Given $F=0$. Find C.

16. What is the temperature when the reading on the two thermometers is the same?

The formula for the area of a rectangle has been given in § 158. Review it.

17. The area of a rectangle of length x and breadth y is A square feet. If each of its dimensions is increased 2 feet, write an expression for its area.



18. The area of the frame of a picture is a. The length of the picture and frame is b and width w. If the frame is c inches wide (see figure) find a in terms of b, w, and c.

275. Summary of Fractional Equations. The subject of equations has been studied in Chapter IV. The present chapter is to fix more firmly in the student's mind the methods used to solve an equation, and to present new applications.

The student will note that by means of transposition of terms the work of solving an equation has been shortened.

An equation may be cleared of fractions by multiplying both its members by a common denominator of its terms.

A new idea has been introduced in connection with extraneous roots. This makes it necessary to check every value obtained for the unknown because some of them may not satisfy the original equation.

Exercise 99 — Review

In the equations given below solve for x and check the result obtained by substituting in the given equation:

1. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{x+2}{2}$

2. $\frac{x-1}{2}-\frac{x-3}{4}=\frac{x-3}{2}$

3. $\frac{1}{3x}+\frac{7}{3}=\frac{1}{x}+\frac{1}{2x}$

4. $6x+4x-13-2x-3=0$

5. $6x-25=7-2x$

6. $3(x-1)^2-3(x^2-1)=x-15$

7. $\frac{x}{6}-\frac{x-2}{5}=\frac{x}{5}-\frac{x-9}{4}$

8. $\frac{4x}{3}+24=2x+6$

9. $\frac{5x-5}{x+1}=3$

10. $\frac{x-2}{5}-\frac{x-3}{4}=\frac{x-7}{10}$

11. $\frac{x}{x+1}=\frac{3x}{x+2}-2$

12. $\frac{36}{x-5}=\frac{45}{x}$

13. $\frac{3x+1}{2}-\frac{2x}{3}=10+\frac{x-1}{6}$

14. $\frac{65}{3x+2}=13$

15. $\dfrac{3x-2}{4}-\dfrac{4-x}{2}=2x-\dfrac{7x-2}{3}$

16. $\dfrac{5x-4}{3}=x+4$

17. $\dfrac{5-3x}{2}=\dfrac{8x-9}{3}$

18. $\dfrac{4x-1}{3}-\dfrac{3}{4}=\dfrac{3x+5}{4}-\dfrac{4-x}{6}$

19. $\dfrac{5+3x}{2}=\dfrac{8x+9}{3}$

20. $\dfrac{12+x}{2x}=\dfrac{12+2x}{3x}$

21. $\dfrac{x+1}{x-4}=\dfrac{x}{x-3}$

22. $\dfrac{65}{3x+2}=\dfrac{13}{3x-2}$

23. $\dfrac{7}{x+3}+\dfrac{1}{x-3}=\dfrac{24}{x^2-9}$

24. $\dfrac{x+3}{x}=\dfrac{x+9}{x+4}$

25. $\dfrac{4}{3x+2}+\dfrac{2}{3x-2}=\dfrac{15x+2}{9x^2-4}$

26. $\dfrac{5}{x+7}-\dfrac{3}{x-5}=\dfrac{2}{x}$

27. $\dfrac{5x-1}{2x+1}+\dfrac{2x+9}{2x-1}=\dfrac{14x^2+49}{4x^2-1}$

28. $\dfrac{9}{5x}-\dfrac{8}{10x-5}=\dfrac{4x-1}{4x^2-1}$

29. $\frac{10x+1}{5}-\frac{4x+7}{6x+11}=\frac{6x-2}{3}$

30. $4(x-a)=3x-5b$

31. $ab+bx=ac+b$

32. $(a+b)x=m-cx$

33. $\frac{x+a}{x-b}=\frac{1}{m}$

34. $(a-b)x=2a-(a+b)x$

35. $\frac{a}{x}-1=\frac{b}{x}-9$

36. $\frac{a-bx}{c}+b=\frac{bc-x}{c}$

37. $\frac{x}{a}+\frac{x}{b}=c$

38. $\frac{1+x}{1-x}=\frac{a}{b}$

39. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=d$

40. $\frac{a}{b+x}-c=d$

41. $\frac{a+x}{a-x}=\frac{a+b}{a-b}$

42. $\frac{a-x}{a}+\frac{b-x}{b}+\frac{c-x}{c}=3$

43. $\frac{ax+b}{ax-b}=\frac{c}{a}$

44. $\frac{x+a}{b}-\frac{b}{a}=\frac{x-b}{a}+\frac{a}{b}$

Exercise 100

Problems Leading to Simple Equations of One Unknown

1. Find a number such that its half, its third, and its fourth shall together make 26.

2. There is a number such that its half less its fourth is 7. What is the number?

3. Two numbers differ by 8, while $\frac{1}{4}$ the smaller equals $\frac{1}{6}$ the larger. Find the numbers.

4. The sum of two numbers is 50 and the smaller number is $\frac{2}{3}$ the larger. What are the numbers?

5. What number is 4 greater than the sum of its half and its third?

6. Ten years ago a man's age was half what it will be 15 years hence. How old is he now?

7. Divide 52 into two parts such that the greater shall be twice the smaller.

8. A man 36 years of age has a son 6 years old. In how many years will the father's age be three times that of his son?

9. If to a certain number $\frac{3}{4}$ of it be added and from the sum 6 be taken, the remainder will be $\frac{3}{2}$ the number. What is the number?

10. A man left $\frac{1}{3}$ his property to his son, $\frac{1}{5}$ to one daughter, $\frac{1}{10}$ to a nephew, and the remainder, which was \$4000, to another daughter. How much property did he have?

11. A can do a piece of work in 15 days and B can do it in 10 days. In how many days can they do it working together?

12. A and B together have \$1000.00. If A gains \$200.00 and B loses \$100.00 they will each have the same sum. How much has each?

13. The length of a room is 4 feet more than its width. If each is increased 2 feet the area of the room will be increased by 52 square feet. What are the dimensions of the room?

14. A can do a piece of work in 6 days. B can do it in 8 days, and C in 16 days. In how many days can they do the work, all working together?

15. A man invests $3000.00, part at 8% per annum and part at 6% per annum. His entire income from the investment is $216.00 per year. How much is invested at each rate?

16. A post is $\frac{1}{5}$ its length in the ground, $\frac{1}{3}$ its length in water, and 14 feet above the water. What is its length?

17. The sum of two numbers is 72. If the greater is divided by the smaller the quotient is 3. Find the numbers.

18. Divide 150 into two parts such that if the smaller is divided by the greater the quotient is $\frac{2}{3}$.

19. If the sides of a square were increased by 4 feet the area would be increased 96 square feet. What is a side of the square?

20. The sum of three consecutive numbers is 36. Find the numbers.

21. A cistern can be filled by one pipe in 5 hours and by another in 6 hours. In how many hours can both together fill it?

22. A man walks up a mountain at the rate of 2 miles per hour and down at the rate of 3 miles per hour. How far did he go if the trip and the return took 10 hours?

23. A train runs 42 miles while a second train runs 57 miles. If the speed of the first train is 5 miles per hour less than that of the second, what were their respective rates?

24. A boy who rows 5 miles per hour in still water finds that he can row to a certain place down stream, and back in 10 hours. The speed of the current is 3 miles per hour. How far down stream is the place?

25. A cistern can be filled by one pipe in 10 hours, by a second pipe in 15 hours, and can be emptied by a third pipe

in 12 hours. In how many hours will the cistern be filled if all three pipes are left open?

26. Three pipes fill a cistern in 5 hours. The first can fill it in 12 hours, the second in 20 hours. In how many hours can the third fill the cistern alone?

27. A can do a piece of work in 8 days. B can do it in 12 days. After A and B work together for 3 days B leaves. How long will it take A to finish the work?

28. A farmer plants $\frac{1}{3}$ his land in cotton, $\frac{1}{4}$ in corn, $\frac{1}{5}$ in peas, leaving the remainder (26 acres) in timber. How many acres in the farm?

29. Six dollars is changed into a number of coins consisting of quarters, dimes, and nickels. There are twice as many dimes as quarters and three times as many nickels as quarters. How many of each are there?

30. A man puts $20,000.00 into two investments. On one he gains 10%; on the other he loses 6%. His total gain is $720.00. What is the amount of each investment?

CHAPTER XIV

SIMULTANEOUS EQUATIONS

DEFINITIONS

276. Degree of a Term. The *degree* of a term is the sum of the exponents of all the letters in it, thus.

$2ax^2$ is of degree 3.

The degree of a term *in a certain letter* is the exponent of that letter, thus,

$2ax^2$ is of degree 2 in x and degree 1 in a.

277. Degree of an Equation. If an equation is rational and integral in certain unknowns (see §§ 202 and 203) then the *degree of the equation* is the degree of the term or terms of highest degree in the equation.

To illustrate, suppose x, y, and z to be the unknowns; show that equation 1 below is of the second degree; equation 2 of the third degree; and equation 3 is of the fifth degree.

1. $12x^2-a^2x+by+c=0$
2. $yz+4x^2-5xyz-x+y^2+8=0$
3. $4x^2y^3-ay^4+by^5+x^5-bxy+7x=0$

If there are any indicated roots, such as $\sqrt{2x-1}$ or $\sqrt[3]{7x^2-13x}$, then the expression "degree of the equation" has no meaning. Also, if the unknowns appear in the denominator, the expression "degree of the equation" has no meaning.

For example, $\sqrt{2x-1}=x+1$.

In this form the words "degree of the equation" have no meaning, but, if both numbers be squared, we get

$$2x-1=x^2+2x+1$$

which is of the second degree in x.

Again, the equation $\dfrac{a}{2x}-2yx^3=2x^3-3y^2$

is *not* of the third degree in x (it is first degree in a and second in y) but, if both members be multiplied by x, we have

$$\frac{a}{2}-2x^4y=2x^4-3xy^2$$

which is of the fourth degree in x, second degree in y, and fifth degree in x and y.

278. Homogeneous Equation. An equation is *homogeneous* in certain letters if every term is of the same degree in those letters.

279. Linear Equation. An equation which has but one unknown in it and in which every term is of the first degree is called a *linear* equation in one unknown.

Nearly all the equations we have studied so far are of this type.

Thus, $2x-3=7-4x$ and $\frac{3}{2}(x-1)+2x=x+2$.

When these equations are solved they have but one root, which is that value of x that will satisfy the equation.

280. Linear Equation, Two Unknowns. An equation which has two unknowns in it and is of the first degree in both the unknowns is called a *linear equation in two unknowns*. Such an equation has no single numerical value of the unknowns for a root such as the linear equation in one unknown had.

Thus, $2x+y=6$ is linear in x and y.

If $y=2$, we have $2x+2=6$, from which we obtain $2x=4$; $x=2$. Thus the pair of values $x=2$, $y=2$ will satisfy the equation. Again, if $y=1$, $2x+1=6$ and $2x=5$ or $x=\frac{5}{2}$. Thus another pair of values is $x=\frac{5}{2}$, $y=1$.

One can readily verify that the following values will also satisfy the equation: ($x=\frac{3}{2}$, $y=3$); ($x=\frac{7}{2}$, $y=-1$); ($x=-1$, $y=+8$); ($x=8$, $y=-10$).

In general, any value whatever may be assigned to either x or y, and a corresponding value of the other can be found such that the pair will satisfy the equation. The student must try this till convinced of its truth.

A solution for x and y of the kind that we had in the linear equation in one unknown is impossible, since there is here an infinite number of pairs of values which satisfy the equation.

Indeed the x may range over the whole set of positive and negative integers and fractions, and the y will change in such a way that the pair will always satisfy the equation.

281. Variable. A number which changes, or is supposed to change, its value in the same discussion is called a *variable.* All numbers that are not variables are *constants.*

In the above equation both x and y are variables. Examples of variables in nature are: the temperature; space traversed by a moving body.

282. Function of a Variable. If two quantities are so related that for every value of one there corresponds one (or more) values of the other, then the second is said to be a *function* of the first.

Thus, in $2x+y=6$ the y is a function of x, for to every value that may be given x there corresponds one value of y. Examples of functions in natural events are: the temperature is a function of the time of day; space traversed by a moving body is a function of the time it has been moving.

SIMULTANEOUS EQUATIONS IN TWO VARIABLES

283. Definition. Two equations are said to be *simultaneous* if the same roots satisfy both equations.

284. Let us now consider together two simultaneous linear equations in x and y:

$$\begin{aligned}(1)\quad 2x+y&=6\\(2)\quad x-y&=9\end{aligned}$$

In each of these equations let us assign to x the series of values indicated in the table and compute the corresponding value of y. The student must verify this table. One of the values of y is wrong in each table. Find it.

For equation (1)

x	y	x	y
−12	30	3	0
− 8	22	4	− 2
− 5	16	5	− 4
− 3	12	6	− 6
− 2	10	7	− 8
− 1	8	9	−12
0	6	12	−18
1	4	15	−20
2	2		

For equation (2)

x	y	x	y
−12	−21	3	− 6
− 8	−17	4	− 5
− 5	−14	5	− 4
− 3	−12	6	− 3
− 2	−11	7	− 2
− 1	−10	9	0
0	− 9	12	3
1	− 8	15	16
2	− 7		

There is just one value of x in the two columns which will make the y of both equations the same. Find it. Verify this value of x and y in both equations to make sure that they are both satisfied by this pair of numbers.

285. Roots of Two Equations. A pair of values which satisfies both of two simultaneous linear equations is called a *set of roots* of the equations.

Exercise 101

Solve the following sets of equations by the method used in § 284:

1. $x+2y=7$
   $3x-2y=5$

2. $2x+3y=22$
   $x-y=1$

3. $5x-2y=21$
   $x-y=6$

4. $3x+15=4y$
   $3y+17=2-3x$

5. $2x-y=3x$
   $4x+2y=3+4y$

6. $x+6y=2x-16$
   $3x-2y=24$

GRAPHICAL REPRESENTATION OF LINEAR EQUATIONS

286. There is another way of showing the various relative values of x and y which is perhaps clearer than that of the table just shown. It is done by making what is known as the *graph of the equation.*

287. Graph of an Equation. By the graph of an equation we mean a picture showing the relative values of the variables.

288. Method of Making the Graph of an Equation. In order to construct such a picture it is necessary to select some point on the paper as a reference point. This point is called the *origin.* Through the origin draw two lines at right angles or perpendicular to each other (§ 138), the one horizontal, the other vertical.

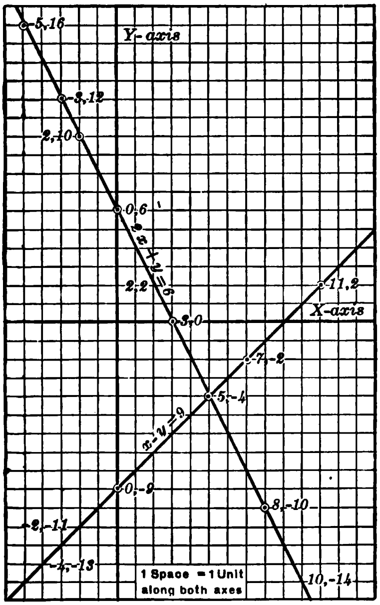
289. Axes. These two lines are lines of reference and are called *axes.*

290. Location of a Point. We can locate any point on the sheet of paper by giving its distance and direction from each

of these axes, so that any two distances fully locate the point; while on the other hand, any point on the sheet of paper has two distances, one measured from each of the axes.

291. Positive and Negative Direction. It is customary to call distances measured to the right from the vertical axis *positive* and distances to the left *negative;* while distances measured upward from the horizontal axis are called *positive* and those measured downward are called *negative.*

It is evident then that each point on the paper corresponds to two distances, one from each of the axes, and the signs of these distances are determined by the directions of the point from the two axes.



In the figure above we plot the set of values shown in the table of § 284 and form the graphs of the two equations given there.

292. Method. Take the side of the small square laid off on the paper as the unit of distance, and locate the various points, the values of x (p. 229) being measured from the vertical axis and the values of y from the horizontal axis. Be careful to make the directions agree with the signs of the values of x and y. Then connect the points so determined by means of straight lines.

293. Axes of X and Y. The horizontal axis is called the *x-axis* and the vertical axis is called the *y-axis*. The two numbers which fix the position of a point are called its *coördinates.*

294. Plotting. The process described above is known as *plotting the graphs* of the given equations. We notice that the points plotted for each equation appear to be in a straight line.* As a matter of fact, these points are in a straight line. For this reason such an equation is called a linear equation. (§ 280.) The reader will notice that these two lines cross at a point whose distances, with their proper signs, from the two axes correspond exactly to the pair of numbers appearing in both tables of § 284; so that the distances from the axes of the crossing point of the two graphs give a pair of values which satisfies both equations.

295. Graphical Solution. This method of determining the pair of values satisfying both equations is called the *graphical method* of solving the two simultaneous linear equations.

296. Graphical Solution of Simultaneous Equations in Two Unknowns. The graphical method of solving two simultaneous equations in two variables, then, is to plot the graphs of the two equations and note the distances of the

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* Since the graph of each equation is a straight line, only two points satisfying a *linear equation* need be determined in order to plot its graph.

crossing points of the graphs from the two axes. The numbers representing these distances are the roots of the two equations. The distance from the y-axis with its proper sign is the value of x, and the distance from the x-axis with its proper sign is the value of y.

Let the student construct the graphs of all the equations in Exercise 101.

SOLUTION OF EQUATIONS BY ELIMINATION

297. Combining Equations. If the two equations can be combined so as to get rid of one of the unknowns then this new equation, containing only one unknown, can be solved by the methods of the previous chapters.

For example, solve $\begin{cases} x+2y=7 \\ 3x-2y=5 \end{cases}$

If these equations be added and the sum of the left members be put equal to the sum of the right members,

$$4x=12$$

Dividing by 4, $\quad x=3$

Substituting this value of x in the first equation

$$3+2y=7$$
$$2y=7-3$$
$$2y=4$$
$$y=2$$

Therefore the set of roots is $x=3$, $y=2$.

Check: $3+4=7$
$9-4=5$

298. Changing the Form. If the two equations are not in such form that immediate addition or subtraction will eliminate one of the unknowns they can always be put in that form by multiplying one or both equations by a properly chosen constant.

For example, (1) $3a+7b=7$
(2) $5a+3b=29$

Multiplying (1) by 5, $15a+35b=35$
Multiplying (2) by 3, $15a+9b=87$

Subtracting $26b=-52$
Dividing by 26 $b=-2$
Substituting in (1) $3a-14=7$
Transposing $3a=21$
Dividing by 3 $a=7$
Therefore the set of roots is $a=7$, $b=-2$.

Check: (1) $3\cdot7+7(-2)=7$
$21-14=7$
(2) $5\cdot7+3(-2)=29$
$35-6=29$

The general rule illustrated by this problem is as follows:

299. Rule. *If immediate addition or subtraction does not eliminate one of the unknowns, multiply one or both equations by numbers so chosen that the coefficients of like letters will be numerically equal.*

If these coefficients have unlike signs, add to eliminate; if their signs are alike, subtract to eliminate.

Solve the resulting equation for one unknown. Substitute its value in one of the original equations and solve for the other unknown.

Check by substituting the pair of values in each equation.

Exercise 102

Solve, plot, check, and compare the set of roots with the points of intersection of the two lines.

1. $x+y=7$
$5x-2y=10$

2. $x-3y=23$
$3y-3x=19$

3. $7x-y=4$
$2x-y=6$

4. $5x+10y=14$
$2x+5y=4$

5. $10a-b=3$
$2a+2b=17$

6. $r-8s=+30$
$r+11s=49$

7. $2x-5y=-16$
$3y+7x=5$

8. $7p-2n=3$
$89=19n-6p$

9. $3s-t=12$
$t-2s=5$

10. $6n=2m+9$
$3m=18n+10$

11. $6c-11d=3d+16$
$3d-4c=6-3c$

12. $31x+144y=275$
$19x-121y=11x-47$

13. $27h-32k=8$
$16k-19h=1$

14. $2r+25t=20$
$63=7r-9t$

ELIMINATION BY SUBSTITUTION

300. The elimination of one of the unknowns may be accomplished by solving one equation for one unknown in terms of the other and substituting this value in the second equation. This will give an equation in one unknown which can be solved. The method of procedure is better explained by an example.

Solve: $\begin{cases} 2x-7y=4 & (1) \\ 3x-2y=3 & (2) \end{cases}$

Solve (1) for x, $x=\dfrac{4+7y}{2}$

Put this value of x in (2) $\dfrac{3(4+7y)}{2}-2y=3$

Clearing of fractions $3(4+7y)-4y=6$

Simplifying $12+21y-4y=6$

Collecting $17y=-6$

$$y=-\frac{6}{17}$$

Hence $$x=\frac{4-\dfrac{42}{17}}{2}=\frac{26}{34}=\frac{13}{17}$$

Check: (1) $\frac{2\cdot13}{17}+\frac{42}{17}=4;\ \frac{26+42}{17}=4;\ 4=4$

(2) $\frac{39}{17}+\frac{12}{17}=\frac{51}{17}=3$

The above example leads us to the following rule:

Rule. *Solve one equation for one of the unknowns. Substitute in the other equation. Solve the resulting equation for the other unknown and check.*

ELIMINATION BY DIVISION

301. The method just described is called *elimination by substitution.*

It is often more convenient to *eliminate by division,* as shown below.

Using the equations given above, write them thus—

$$(1)\quad 2x=7y+4$$
$$(2)\quad 3x=2y+3$$

Dividing, $\frac{2}{3}=\frac{7y+4}{2y+3}$

Clearing, $4y+6=21y+12$

Combining terms, $-6=17y$

Solving for y, $y=-\frac{6}{17}$

From equation (1), $2x=-\frac{42}{17}+4=\frac{26}{17}$

Solving for x, $x=\frac{13}{17}$

Check as above.

Rule. *Transform the given equations so the x-terms (or the y-terms) stand alone in one member. Divide one of the equations by the other, and solve the resulting equation for the unknown that is not eliminated. Find the other unknown and check as above.*

Exercise 103

Solve by substitution or by division:

1. $2x-4y=16$
   $3x+y=7-y$

2. $2x+12y=26$
   $6x-7y=-8$

3. $x+4y=4$
   $2x+2y=5$

4. $9x-y=5$
   $15x+2y=23$

5. $7h-k=1$
   $k-6h=0$

6. $2x-y=36$
   $x-y=29$

7. $3s+12=3+2t$
   $2t=s+1$

8. $5x-2y=y$
   $3x-6y=0$

9. $\frac{3}{4}x+4y=\dfrac{-x}{4}+4$
   $\frac{3}{2}x+\frac{1}{2}y=5-\frac{1}{2}x-\frac{3}{2}y$

10. $\dfrac{2}{10x-4y}=\dfrac{1}{y}$
    $\dfrac{1}{3(x-2y)}=1$

11. $6x+5y=-13$
    $2y-4x=9$

12. $64x-81y=2$
    $16x-8y=17$

13. $2+3x=13y+11$
    $6x-4=4y-18$

14. $34x+70y=2y+102$
    $5x-8y=4$

15. $\dfrac{r+2s}{3}=\dfrac{3}{2}(r-3s)$
    $r-\frac{2}{3}s=0$

SIMULTANEOUS EQUATIONS CONTAINING FRACTIONS

302. In some of the exercises in the last few sections fractional coefficients were used. Any equation is linear, provided it is of first degree after it has been cleared of fractions and the unknowns collected. In the following exercises the equations should first be cleared of fractions by multiplying both members by the l. c. m. of the denominators and then solved by the methods of §§ 299, 300, and 301.

Exercise 104

Solve and check the following pairs of equations:

1. $\frac{2}{5}x+2y=4$
   $\frac{3x-10y}{5}=1$

2. $\frac{2}{9}x-\frac{3}{2}y=6$
   $2x-\frac{9}{5}y=-4\frac{1}{5}$

3. $.4x+2y=4$
   $.6x-2y=1$

4. $\frac{x-1}{y-1}=\frac{3}{-2}$
   $\frac{x+3}{4}=\frac{-y+7}{3}$

5. $\frac{2x-7}{4y+1}=\frac{-7}{3}$
   $\frac{x+1\frac{1}{3}}{y-1.5}=\frac{1\frac{3}{5}}{2}$

6. $\frac{x-4}{y+3}=\frac{4}{3}$
   $\frac{x+2}{y+5}=\frac{3}{5}$

7. $\frac{x-a}{y+b}=\frac{a}{b}$
   $\frac{x+a}{y-b}=\frac{a+1}{b}$

8. $\frac{4.5x-1}{.3y+2}=.75$
   $\frac{x+y}{x-y}=.6$

9. $\frac{1}{x}+\frac{1}{y}=\frac{1}{xy}$
   $\frac{2}{x}-\frac{3}{y}=\frac{x+y}{xy}$

   Use $\frac{1}{x}$ and $\frac{1}{y}$ as unknowns.

10. $\frac{2x+y}{3}-\frac{3x-2y}{4}=2.5$
    $\frac{x+y}{2}+\frac{3x-1}{4}=\frac{1}{3}$

11. $\frac{1}{x}+\frac{4}{y}=6$
    $\frac{3}{2x}+\frac{1}{y}=-2$

    Use $\frac{1}{x}$ and $\frac{1}{y}$ as unknowns.

12. $\frac{3a-8}{9}-\frac{5b-3}{2}=31$
    $\frac{7a-1}{5}+\frac{3b+6}{10}=-28$

13. $\frac{x}{a}-\frac{x}{2a}+\frac{y}{b}=1$
    $bx-3ay=ab$

14. $\frac{7t+6}{5}-\frac{7r-3}{3}=4$
    $\frac{2t-7}{2}+\frac{r-4}{4}=2.5$

303. Problems Involving Two Unknowns. Many problems may be solved by using either one or two unknowns. Sometimes one way is to be preferred and sometimes the other, thus,

Example. A line 10 feet long is divided into two parts such that twice one part is three times the other. How long is each part?

Solution by two unknowns.

Let x = one part.
Let y = the other part.
Then $x+y$ = the sum of the parts.
But 10 = the sum of the parts.
(1) Therefore, $x+y=10$.
$2x$ = twice one part.
$3y$ = three times the other.
(2) Hence, $2x=3y$.
Solving these two equations simultaneously, from (2)

$$x=\frac{3y}{2}$$

Substituting in (1)

$$\frac{3y}{2}+y=10$$
$$3y+2y=20$$
$$5y=20$$
$$y=4$$
$$x=6$$

Check: $4+6=10$ and
$2\times6=3\times4$

Solution by one unknown.

Let x = one part.
Then $10-x$ = the other part.
$2x$ = twice one part.
$3(10-x)$ = three times the other part.

Therefore $2x=3(10-x)$
$$2x=30-3x$$
Transposing $2x+3x=30$
$$5x=30$$
$$x=6$$
$$10-x=4$$

Check: $6+4=10$

As a rule the solution by two unknowns is longer, but the equations are more easily formed than by the method of a single unknown. There are also some problems which cannot be easily solved by one unknown; hence it is desirable for the student to learn how to manage two unknowns in the solution of problems.

Exercise 105

1. A line 15 feet long is divided into two parts such that one part is $\frac{2}{3}$ the other. Find both parts.

2. A rectangle is $\frac{1}{2}$ as wide as it is long. Its total perimeter is 12 inches. Find the length and breadth.

3. A sum of money is divided between two boys so that one receives $1 more than twice the amount the other receives. The difference between what they receive is $9. Find the amount each receives and the total amount.

4. A sum of money consisting of nickels and dimes is divided so that there is as much in nickels as dimes. If the number of dimes is increased so that there are as many dimes as nickels, the sum would be increased one dollar. How many nickels and dimes are there?

5. The length of a rectangle exceeds the width by $2\frac{3}{4}$ feet. The perimeter exceeds twice the width by $6\frac{1}{2}$ feet. Find the dimensions.

6. The width of a rectangle is to the length as 3 : 5. The difference between the length and width is .3 ft. Find the dimensions.

7. Two boys weighing respectively 45 and 72 lb. balance on a seesaw. If the 45 lb. boy is replaced by a 91 lb. boy the fulcrum must be moved 3 ft. Find the length of the seesaw.

8. Two boys balance on a seesaw when the distances from the fulcrum are 5 ft. and 9 ft. If the distances were made equal one boy must be replaced by another 40 lb. heavier. Find the weights of the three boys.

9. The weights of two boys are in the ratio of 7 : 5. If the length of the seesaw on which they balance is 14 ft., find the position of the fulcrum. Can you find also their weights?

10. Two waterpower plants deliver horsepowers which are in the ratio 3 : 5. The height of the dam in one is 36 ft. and in the other 14 ft. If the second stream flows 54,000 cu. ft.

per minute find the stream flow of the first in cubic feet per minute.

11. The power delivered by one hydraulic plant is three times that of another. The stream flow in the first is 11,000 cu. ft. per minute and in the second 36,700. If the sum of the heights of the dams is 110 ft., find approximately the height of the dam and the horsepower in each.

12. A sum of money amounts to a dollars in 1 yr. 8 mo. at 6%. The same sum amounts to $5 more in 1 yr. 4 mo. at 8%. Find the sum and the amount.

13. A sum of $950 amounts to a dollars in time t at 5%. It amounts to $48.26 less in the same time at 4%. Find the amount and the time.

14. A sum of $390 amounts to a dollars in $3\frac{1}{3}$ years. The same sum amounts to $15.60 more in 4 years. Find the rate and the amount.

15. A number is composed of two digits. The sum of the digits is 13. The difference between four times the tens' digit and three times the units' digit is 3. Find the number.

16. The sum of the two digits of a certain number is 54 less than the number. If the digits are interchanged the number is increased by 27. Find the number.

17. A can do $\frac{3}{4}$ as much work per day as B. Together they can do a certain piece of work in 12 days. How long does it take each to do it working alone?

18. A farmer has 80 horses and cows. They are worth $5660, counting the cows at $40 each and the horses at $100 each. How many of each are there?

19. A and B can do a piece of work in 6 days. After working together for 2 days, A quit and B finished the work alone in 10 days. How many days would each require for the work?

20. There is a certain fraction such that if 3 be added to

each of its terms the result will be $\frac{2}{3}$; while if 3 be subtracted from each of its terms the result will be $\frac{1}{3}$. What is the fraction?

21. A number is composed of two digits such that their sum is 4 times their difference. If 18 be subtracted from the number the order of the digits will be reversed. Find the number.

22. Determine values of x and y satisfying the equations

$$x-\frac{2y-x}{23-x}=20+\frac{2x-59}{2}$$

$$y-\frac{y-3}{x-18}=30-\frac{73-3y}{3}$$

23. A note at simple interest amounted to $660.00 in two years and in five years to $750.00. What was the face of the note and the rate?

24. One angle of a triangle is 50 degrees. Of the other two angles the smaller is 5 degrees more than $\frac{1}{4}$ of the greater. Find the two angles.

25. A requires 4 hours more than B to walk 30 miles, but if A would double his rate then he could walk the distance in one hour less than B. How fast does each walk?

26. If the numerator of a fraction is doubled and the denominator increased by 3 the fraction reduces to $\frac{2}{3}$. If the numerator is increased by 2 and the denominator is quadrupled the fraction reduces to $\frac{1}{4}$. What is the fraction?

27. If a number is divided by the difference between its digits the quotient is 16 with remainder 3. Twice the units' digit is 2 less than the tens' digit. What is the number?

28. Find the values of x and y that will satisfy the following equations:

$$\frac{5}{x-1}-\frac{4}{y-2}=12 \quad \text{and} \quad \frac{4}{x-1}+\frac{3}{y-2}=22$$

Note. The equations may be combined without clearing of fractions.

29. Find values of x and y to satisfy these equations:

$$\frac{2}{xy}+\frac{2}{y}=1 \text{ and } \frac{3}{xy}-\frac{7}{y}=\frac{-11}{3}$$

30. A pound of tea and 10 pounds of sugar cost $1.40. If tea were to rise in value 25% and sugar 15% they would cost $1.66. What is the cost per pound of each?

31. Admission to a show was 10 cents for adults and 5 cents for children. If the sale of 100 tickets brought $8.55, how many of each were sold?

32. A grocer mixes coffee worth 15 cents a pound with coffee worth 35 cents a pound so that 100 pounds of the mixture is worth $22.40. How many pounds of each kind are there?

33. If the greater of two numbers is divided by the smaller the quotient is 3 and the remainder 1, but when 11 is added to the sum of the numbers the quotient obtained by dividing this sum by the greater is 2. What are the numbers?

34. Find values of x and y that will satisfy the equations:

$$\frac{5}{x+2y}=\frac{7}{2x+y} \text{ and } \frac{7}{3x-2}=\frac{5}{6y-7}$$

35. Two hunters together killed 36 birds. $\frac{3}{5}$ the number killed by the first hunter is 4 more than half the number killed by the second. How many birds did each kill?

36. A man paid 3 men and 5 boys $40.50 for one day and later at the same rates paid 6 men and 3 boys $49.50 for one day. What was paid each man and each boy per day?

37. In a meeting of 250 persons, all voting, a motion was carried by a majority of 34. How was the vote divided?

38. Find two fractions with the numerators 2 and 3 respectively whose sum is $\frac{41}{63}$ and such that if the numerators are interchanged their sum is $\frac{13}{21}$.

SIMULTANEOUS EQUATIONS IN THREE UNKNOWNS

304. Equations with Three Unknowns. Sometimes problems arise in which there are three or more unknown quantities. We will confine our attention to the case of linear equations in three unknown quantities. To be prepared to handle such problems we must first make a study of the methods of solving for three unknowns. One of the chief advantages of the algebraic method of solving a pair of simultaneous equations in two unknowns, is that it is easily extended to three, four, five, or even more equations in as many unknowns.

Example. Three business men, in discussing the number of clerks in their stores, found that the total number was 32, while twice the number the first had plus the number the second and third had equaled 52. If the number the second had were doubled and added to the number in the stores of the first and third the result equaled 40. How many had each?

Let x stand for the number the first had.
Let y stand for the number the second had.
Let z stand for the number the third had.

Then
$$x+y+z=32 \quad (1)$$
$$2x+y+z=52 \quad (2)$$
$$x+2y+z=40 \quad (3)$$

Subtracting (1) from (2), we get $x=20$.

Subtracting (1) from (3), we get $y=8$.

Substituting these values in (1), $20+8+z=32$.

Transposing, $z=4$

Check:
(1) $20+8+4=32$
(2) $40+8+4=52$
(3) $20+16+4=40$

It does not often happen that a subtraction such as the first above will get rid of two variables at once. In any case one can always multiply two of the equations by such numbers that a subtraction (or addition) will eliminate one

unknown. Then two other equations can be combined so as to eliminate the same unknown, leaving two equations and two unknowns in them which can be found by the methods of the preceding articles.

Example.	(1)	$2x-3y+z=1$
	(2)	$x-2y+5z=-2$
	(3)	$3x+y-3z=4$
Rewriting (1),		$2x-3y+z=1$
Multiplying (2) by 2,		$2x-4y+10z=-4$
Subtracting (4),		$y-9z=5$
Rewriting (3),		$3x+y-3z=4$
Multiplying (2) by 3,		$3x-6y+15z=-6$
Subtracting (5),		$7y-18z=10$
Multiplying (4) by 7,		$7y-63z=35$
Subtracting,		$45z=-25$
Dividing,		$z=-\frac{5}{9}$
From (4),		$y=9z+5=-5+5=0$
From (2),		$x=2y-5z-2=0+\frac{25}{9}-2=\frac{25-18}{9}=\frac{7}{9}$
Check:	(1)	$\frac{14}{9}-\frac{5}{9}=1$
	(2)	$\frac{7}{9}-\frac{25}{9}=-\frac{18}{9}=-2$
	(3)	$\frac{21}{9}+\frac{15}{9}=\frac{36}{9}=4$

It is necessary to check in all three equations to be sure of accuracy. For instance, $x=1$, $y=\frac{2}{13}$, $z=\frac{7}{13}$ will check in the first two equations but not in the third. Why not?

Exercise 106

Solve for all the unknowns and check in each equation:

1. $x+y+z=4$
   $x-y+z=2$
   $2x+y+z=1$

2. $3x-y-z=1$
   $x+2y-2z=-3$
   $x-3y+z=2$

3. $4x+5y+z=7$
$3x+4y+5z=6$
$2x+3y+4z=5$

4. $x+2y+3z=4$
$y+2z=3$
$x+z=-2$

5. $2x+y=-1$
$3x+2y+z=\frac{1}{2}$
$\frac{2}{3}x+z+\frac{1}{5}=y+4$

6. $18x+6\frac{1}{2}=\dfrac{6y+3}{2}$
$7\frac{1}{4}x+2z=\dfrac{4y+x}{4}$
$x-y=10$

305. Summary of Simultaneous Equations. There are four ways of solving a pair of simultaneous equations, viz.:

A. Eliminate one of the unknowns by addition or subtraction, and solve the resulting equation for the other unknown.

B. Eliminate one of the unknowns by division, and solve the resulting equation for the other unknown.

C. Solve one equation for one unknown in terms of the other. Substitute this result in the other equation. This will give an equation in one unknown which can be solved by the methods of the previous chapter.

D. Plot the graphs of each of the two equations and note the distances of the point of intersection from the axes, giving the proper sign to these distances.

There are other methods but the four given are sufficient for the solution of any pair of simultaneous equations.

In the case of three equations and three unknowns only one method was given. It consisted of the elimination of two unknowns which gave an equation in one unknown. Again, there are other methods but they need not be discussed here.

The student is especially urged to master thoroughly the solution of literal equations. This is important because it *combines in one* solution all possible cases of that type.

Exercise 107—Problems in Three Unknowns

1. A line 21 feet long is divided into three parts. The first and second together make 6 feet more than 4 times the third. The sum of the first and third equals one less than the second. Find all three parts.

2. A man invested $16,000 in stocks and bonds. There were three blocks of them which at $4\frac{1}{2}\%$, 5%, and 6% yielded $800 per year. If the second block was twice the first plus $1000, find all three.

3. A gentleman in his will divided his estate of 18,000 acres among his 3 children in such a way that the eldest lacked 500 acres of getting twice as much as the second, and the second lacked 600 acres of getting twice as much as the third. How many acres did each get?

4. An issue of $125,000 of bonds were sold at three prices, at a premium of 2%, at par, and at a discount of 1%. The total amount received was $125,550. If twice the amount sold at a discount lacked $10,000 of being the amount sold at a premium, find each amount.

5. Three boys balance on a seesaw, two on one end when the distances from the fulcrum are 7 and 5 feet. If the fulcrum be placed in the middle the boy sitting by himself must take a 20 lb. stone in his lap to balance. If one boy is taken off, the other two balance when the fulcrum is 8 feet from one end. How heavy is each boy?

6. A, B, and C working together do a piece of work in 6 days. A and C can do it in 8 days, B and C in 16 days. In how many days can each do it alone?

7. Find three numbers whose sum is 15, if the sum of the first and three times the second is 11, and the sum of the second and the third less the first is 9.

8. Divide 600 into three parts such that the sum of $\frac{1}{2}$ the

first, $\frac{1}{4}$ the second, and the third shall be 250; and the first less $\frac{1}{2}$ the sum of the other two shall be 150.

9. A number is composed of three digits of which the hundreds digit is $\frac{2}{3}$ the sum of the other two; the sum of the hundreds digit and the units digit is double the ten's digit, while the sum of the three digits is 15. What is the number?

LITERAL EQUATIONS

306. We have already found in many places that the ability to handle equations with literal coefficients is a means of saving a great deal of labor in numerical computation by enabling us to develop formulas that hold in every possible case. Below we give a list of problems which are arranged with the idea of giving greater skill in handling equations with literal coefficients:

Exercise 108

1. $x+ay=1$
   $2x-y=a$

2. $3ax+ay=2ax-1$
   $x-\dfrac{y}{a}=7$

3. $\dfrac{x}{a}+\dfrac{y}{b}=1$
   $2bx+y=ab$

4. $\dfrac{A+B}{2}=C-180$
   $\dfrac{A-B}{2}=65$
   $\dfrac{A}{2}+C=91$

5. $\dfrac{x}{a^2-b^2}+\dfrac{y}{a-b}-3=\dfrac{1}{a^2-b^2}$
   $3x-ay=-by+3$

6. $ax+by=1$
   $2x-3by=a^2$

7. $2a=3p(1+.12)$
   $a=p(1+.06)+1$

8. Solve for t_1 and t_2:
   $v_1t_1=3v_2t_1+3$
   $v_1t_2=v_2t_2+16$

9. $a_1x+b_1y=1$
   $6x-3y=b_2$

10. $a_1x+b_1y=3$
    $a_2x+b_2y=c$

11. $a_1x+b_1y=c_1$
    $a_2x+b_2y=c_2$

CHAPTER XV

SQUARE ROOTS AND RADICALS

DEFINITIONS

307. In §§ 206 to 208 we defined square root and cube root and showed how to find the square and cube roots of monomials. The student should review these definitions and exercises. His progress in this chapter is impossible unless he has learned the meaning of the phrase "square root of an expression" and knows how to find the square root of monomials rapidly.

308. By the expression **root of a quantity** is meant one of its equal factors.

309. The symbol indicating that a root of a quantity is to be found is the sign $\sqrt{\ }$ commonly known as the **radical sign** or more simply as the **radical.**

310. A number written in the opening of the radical sign is called the **index** of the root. It denotes the number of equal factors in the quantity of which one is to be determined. If no index is written 2 is understood as the index.

Examples. $\sqrt[3]{8}$ (read the cube root of 8) is one of the three equal factors of 8. $\sqrt{16}$ (read the square root of 16) is one of the two equal factors of 16.

The square root of a polynomial may sometimes be found by inspection. For example, since

$$(a+b)^2=a^2+2ab+b^2$$

it follows that

$$\sqrt{a^2+2ab+b^2}=+(a+b) \text{ or } -(a+b)$$

Again, $(x+y+1)^2=x^2+y^2+1+2xy+2x+2y$

Therefore $\sqrt{x^2+y^2+1+2xy+2x+2y}=\pm(x+y+1)$

Exercise 109

311. Find the square roots of the following by inspection. Check by multiplying.

1. $x^2+y^2+1-2xy+2x-2y$
2. $4x^2+4ax+a^2$
3. $4x^2+4ax+a^2+b^2-4bx-2ab$
4. $9a^2+b^2+1+6ab-6a-2b$
5. $x^2+4y^2+1+4xy-2x-4y$
6. $9x^2+4y^2+16-12xy+24x-16y$
7. $x^2+y^2+z^2+1+2xy+2xz+2x+2yz+2y+2z$
8. $4x^2+y^2+a^2+1-4xy+4ax-4x-2ay+2y-2a$

SQUARE ROOTS OF POLYNOMIALS

312. The square root of any polynomial may be found by inspection provided the terms do not combine or cancel.

$$(x^2-2x+4)^2=x^4+4x^2+16-4x^3+8x^2-16x=$$
$$x^4-4x^3+12x^2-16x+16$$

One would not usually suspect that this polynomial of five terms had a square root of three terms. For this reason we need to obtain a method of extracting the square root of a polynomial.

313. Extraction of Square Root. We now undertake to work out the rules for extraction of the square root of any polynomial.

Consider the following identity:

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$$

We have a decided advantage in this case because we

know the answer already. The work is arranged as in long division.

		Polynomial	Root
		$a^2+2ab+2ac+b^2+c^2+2bc$	$a+b+c$
		a^2	
Trial divisor	$2a$	$+2ab+2ac+b^2+c^2+2bc$	1st remainder
	$+b$		
Complete divisor	$2a+b$	$2ab \quad +b^2$	
2d trial divisor	$2a+2b$	$2ac \quad +c^2+2bc$	2d remainder
	$+c$		
Complete 2d divisor	$2a+2b+c$	$2ac \quad +c^2+2bc$	
		0	3d remainder

314. The explanation of the above work is as follows: Arrange the polynomial according to descending powers of some letter, as a. For the first term of the result extract the square root of first term of the polynomial. Subtract the square of this term. For a trial divisor double the root already found. Divide the first term of the first remainder by this trial divisor for the second term of the root. Add this quotient to the trial divisor for a complete divisor. Multiply the complete divisor by the second term of the root and subtract. Repeat this process till the last remainder is zero. Check by squaring the root found.

Exercise 110

Find the square root of:

1. $x^2+y^2+1-2xy+2x-2y$
2. $4a^4+1+4a^3-3a^2-2a$
3. $1+8a^2+4a^4-4a-8a^3$
4. $x^4+4x^3+6x^2+4x+1$
5. $x^4+2x^3-3x^2-4x+4$
6. $c^4-c^2+2c^3-2c+1$

7. $x^4-x^3-\dfrac{15x^2}{4}+2x+4$

8. $25r^2+\dfrac{t^2}{100}+\dfrac{1}{4}+rt-5r-\dfrac{t}{10}$

9. $100r^4+2r^3+\dfrac{2001r^2}{100}+\dfrac{r}{5}+1$

10. $x^4+y^4+x^2(1+2y^2+2y)+y^2(1+2x+2y)+2x(x^2+y)$

11. $x^4+y^2+3x^2+2xy(x-1)-2x(x^2+1)+2y+1$

12. $a^6+\dfrac{9a^4}{4}+a^5-a^3-2a+1$

13. $4a^6-12a^5+13a^4-14a^3+13a^2-4a+4$

14. $\dfrac{m^2}{n^2}+\dfrac{n^2}{m^2}+\dfrac{17}{4}-\dfrac{5m}{n}+\dfrac{5n}{m}$

SQUARE ROOT OF ARITHMETICAL NUMBERS

315. Since $1^2=1$ and $9^2=81$, it follows that the square root of a number of one or two digits has only one digit. Since $10^2=100$ and $99^2=9801$, it follows that the square root of a number of three or four digits has two digits in it. Similarly, since $(.1)^2=.01$ and $(.9)^2=.81$, and since $(.01)^2=.0001$ and $(.99)^2=.9801$, it follows that a decimal fraction of one or two decimal places has only one decimal place in its square root, while one of three or four places has two places in its square root. These results can easily be extended to numbers of five or more digits and to decimals of five or more places. This gives the following rule:

316. Rule. *Begin at the decimal point and mark the number into groups of two digits each. Take the largest square in the*

left hand group as the first digit of the root. Then proceed exactly as in the case of algebraic expressions.

For convenience, so that the student may compare them, we arrange below an arithmetical and an algebraic example in parallel columns.

	$a+b+c$		$400+10+3$
	$a^2+2ab+2ac+b^2+2bc+c^2$		$17'05'69$
	a^2		160000
$2a$	$2ab+2ac+b^2+2bc+c^2$	800	10569
$+b$		10	
$2a+b$	$2ab \quad +b^2$	810	8100
$2a+2b$	$2ac \quad +2bc+c^2$	820	2469
$+c$		3	
$2a+2b+c$	$2ac \quad +2bc+c^2$	823	2469
	0		0

The number is divided into three groups of two digits each, as shown. The largest square in 170,000 is $(400)^2$. After subtracting $(400)^2$ from the radicand* the remainder is 10569. Then we obtain the trial divisor by doubling the first term (400) of the root, which gives 800 in the arithmetic example corresponding to $2a$ in the algebraic example. The second term is obtained by dividing the trial divisor into the first remainder. This giving 10, the complete divisor is obtained by adding 10 to 800 just as b was added to $2a$ for the complete divisor. Multiply 810 by 10 and subtract. We obtain a second trial divisor by doubling 410, then proceed as before. Thus the process of finding arithmetical square root is exactly the same as the algebraic process except for pointing off and finding the first digit.

In practice, the work of extracting roots is somewhat abridged, as shown by the example on page 254.

* *Radicand* is the number whose root is to be found.

Example. **Find correct to three decimal places the square root of 829.667.**

$$
\begin{array}{r|l}
 & 28.803+ \\
 & 8'29.66'70'00' \\
 & 4 \\ \hline
48 & 429 \\
 & 384 \\ \hline
568 & 4566 \\
 & 4544 \\ \hline
57603 & 227000 \\
 & 172809 \\ \hline
\end{array}
$$

The work can be carried any number of steps by simply annexing zeros in groups of two.

Exercise 111

Extract the square root of the numbers below. In case the number is not a square carry the work to two decimal places.

1. 294,849
2. 38,416
3. 4,726,276
4. 599,076
5. 231,745.61
6. 87.4225
7. 917.43496
8. .9801
9. 1.674
10. 223,729
11. 291,475
12. .0037
13. .0463
14. $\frac{16}{25}$
15. $\frac{81}{144}$
16. $16\frac{2}{3}$

Note that to square a fraction one squares the numerator and the denominator separately; hence to extract the square root of a fraction one extracts the square root of the numerator and the denominator separately. If either of these is not a perfect square it is better to reduce to a decimal before extracting the root. This applies also to mixed numbers.

For example: $\sqrt{6\frac{4}{7}} = \sqrt{6.571428}$

```
            2.563
          ________
          6.57'14'28
          4
          ________
45       |2.57
         |2.25
          ________
506       |3214
          |3036
          ________
5123       |17828
           |15369
           ________
```

17. $1\frac{6}{7}$ **18.** $\frac{13}{5}$ **19.** $\frac{3}{22}$ **20.** $121\frac{112}{41}$

Simplify by reducing to decimals and adding:

21. $\sqrt{43}+\sqrt{\frac{1}{3}}$

22. $\sqrt{2}+\frac{1}{\sqrt{2}}$

23. $\sqrt{46}+\sqrt{105}-\sqrt{2}$

24. $\frac{1}{\sqrt{2}}-\frac{\sqrt{2}}{2}$

SIMPLIFICATION OF RADICALS

317. Simplest Form. An expression involving radicals is said to be in its *simplest form* if there are no radicals in the denominator and if the radicand has no factors which are perfect powers.

318. It appeared in Chapter X that *the square root of the product of several numbers equals the product of the square roots of the factors.*

For example, $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{x^2y} = x\sqrt{y}$

This principle is often useful in saving labor in computation. For example, suppose the square root of 2 has been found to be 1.4142; then it is not necessary to extract the square root of 8, for $\sqrt{8} = \sqrt{4\cdot 2} = 2\sqrt{2} = 2(1.4142) = 2.8284$.

319. A radical may be removed from the denominator by multiplying numerator and denominator by such a

radical that the denominator becomes free from radicals, thus,

$$\frac{6}{\sqrt{2}}=\frac{6\sqrt{2}}{\sqrt{2}\sqrt{2}}=\frac{6\sqrt{2}}{2}=3\sqrt{2}$$

From what has just been said, we have the following rule:

320. Rule. *To simplify an expression involving multiplication and division of radicals of index 2*; (1) *factor the expression and extract the square root of all factors that are squarcs;* (2) *multiply the numerator and denominator by the radical part of the denominator.*

Exercise 112

Reduce to simplest form:

1. $\sqrt{18}$
2. $\sqrt{343}$
3. $\sqrt{62.5}$
4. $\dfrac{\sqrt{20}}{34}$
5. $\dfrac{2\sqrt{24}}{\sqrt{125}}$
6. $62\sqrt{98}$
7. $\sqrt{x^2y}$
8. $\sqrt{a^3b^2}$
9. $\sqrt{32x^2y^4}$
10. $\sqrt{162a^2c^5}$
11. $\sqrt{x(1-x)^2}$
12. $\sqrt{a(1-a)(1-a^2)}$
13. $\sqrt{x^3+2x^2y+xy^2}$
14. $\sqrt{8a}+\sqrt{12a}$
15. $\dfrac{1}{\sqrt{2}}$
16. $2\sqrt{\frac{1}{2}}$
17. $\dfrac{8}{\sqrt{8}}$
18. $\dfrac{17}{\sqrt{18}}$

ADDITION AND SUBTRACTION OF RADICALS

321. Two or more expressions involving radicals can be added or subtracted in any case by prefixing the proper signs,

but two or more terms cannot be combined into one unless they have the same *index* for the radical and the same *radicand.*

For example, $2\sqrt{3}+4\sqrt{3}=6\sqrt{3}$, but $2\sqrt{2}+3\sqrt{3}$ cannot be further combined.

Again, $\sqrt{2}+2\sqrt[3]{2}$ cannot be combined into one term because the indices of the radicals are different. However $3\sqrt{2}$ and $2\sqrt{8}$ can be combined since $2\sqrt{8}=4\sqrt{2}$. Hence $3\sqrt{2}+2\sqrt{8}=3\sqrt{2}+4\sqrt{2}=7\sqrt{2}$

The above examples illustrate the following rule:

322. Rule. *To add or subtract two or more radicals reduce to simplest form and combine those which have the same index and the same radicand.*

Exercise 113

Simplify and collect:

1. $\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}$

Solution: $\sqrt{\frac{1}{2}}-\frac{3}{2}\sqrt{2}=\frac{1}{\sqrt{2}}-\frac{3}{2}\sqrt{2}=\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}-\frac{3}{2}\sqrt{2}=\frac{1}{2}\sqrt{2}-$
$\frac{3}{2}\sqrt{2}=(\frac{1}{2}-\frac{3}{2})\sqrt{2}=-\sqrt{2}$

2. $\frac{1}{2}\sqrt{54}-2\sqrt{24}$
3. $\sqrt{90}-3\sqrt{160}$
4. $3\sqrt{\frac{1}{3}}-3\sqrt{\frac{4}{3}}+\sqrt{\frac{4}{6}}$
5. $\sqrt{2a^2}+\sqrt{8a^2}$
6. $\sqrt{a^2x}+\sqrt{4a^2x^3}$
7. $2\sqrt[3]{8a^2x^3}-\sqrt[3]{3a^2x^6}$
8. $\sqrt{9a^3b^3}+2a^2b^2\sqrt{\frac{1}{ab}}+ab\sqrt{\frac{(a-b)^2}{ab}}$
9. $\frac{1}{2}\sqrt[3]{80(a-2)^3}-\sqrt[3]{270}$
10. $\sqrt{x^3-4x^2+4x}+\sqrt{x^3}-\frac{4}{x}\sqrt{x^5}$

MULTIPLICATION AND DIVISION OF RADICALS

323. This section deals only with square roots.

Since $\sqrt{ab} = \sqrt{a}\sqrt{b}$ we have only to read this equation backward to obtain the rule for multiplying radicals.

324. Rule. *To multiply two radicals of the same index put the product of the radicands under the radical sign.*

Since $\sqrt{a} \div \sqrt{b} = \dfrac{\sqrt{a}}{\sqrt{b}}$ we have only to read this equation backward to obtain the rule for dividing radicals.

325. Rule. *To divide one radical by another of the same index put the quotient of the radicands under the radical sign.*

Exercise 114

Perform the indicated multiplications and divisions, and simplify:

1. $\sqrt{7} \cdot \sqrt{9}$
2. $\sqrt{12} \cdot \sqrt{14}$
3. $\sqrt{18} \cdot \sqrt{3}$
4. $\sqrt{20x} \cdot \sqrt{5x}$
5. $\sqrt{a^2b} \cdot \sqrt{b^3}$
6. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{8}}$
7. $(\sqrt{2} - \sqrt{5})(\sqrt{2} + \sqrt{5})$

   Solution: $\sqrt{2} - \sqrt{5}$

   $\sqrt{2} + \sqrt{5}$

   $\sqrt{4} - \sqrt{2}\sqrt{5} + \sqrt{2}\sqrt{5} - \sqrt{25} = 2 - 5 = -3$
8. $(\sqrt{2} + \sqrt{5})^2$
9. $(3 - \sqrt{2})^2$

10. $\sqrt{2}(3\sqrt{2}-2\sqrt{3}-\sqrt{27})$

11. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$

12. $(\sqrt{a^2b}-\sqrt{ab^2})(\sqrt{ab^3}+\sqrt{a^3b^3})$

13. $(\sqrt{5a}-\sqrt{10a^2})\sqrt{5a}$

14. $\dfrac{\sqrt{10}}{\sqrt{2}}$

15. $\dfrac{13\sqrt{12}}{26\sqrt{6}}$

16. $\dfrac{\sqrt{5a^3}-10\sqrt{a}}{\sqrt{5a}}$

17. $\dfrac{(\sqrt{5}+\sqrt{3})(2\sqrt{5}-\sqrt{3})}{6\sqrt{15}}$

18. $\dfrac{\sqrt{\dfrac{a^2}{27a}}-\sqrt{\dfrac{a}{3}}}{\dfrac{16\sqrt{a}\sqrt{3}}{3}}$

326. Summary of Chapter XV. The student who has studied the extraction of roots in arithmetic will recall the difficulty of remembering the rules. These rules have been explained in § 316 in such a way that one can extract square roots by reversing the formula $(a+b)^2=a^2+2ab+b^2$. A similar method is applicable to the extraction of the cube root.

In addition to square roots of arithmetical numbers a process was given § 313 for finding square roots of algebraic polynomials provided these are squares.

In case no exact root of a number can be found then a new kind of number arises, called an *irrational* number. We

have assumed that the laws governing operations with these numbers are the same as the laws of operation on the numbers of arithmetic. From these assumptions we have learned to add, subtract, multiply, and divide irrational numbers. Having learned to manage expressions involving radicals we turn our attention to a few applications.

SOLUTION OF EQUATIONS BY SQUARE ROOT

327. So far we have considered only linear equations. Now we are able to solve a few simple equations of the second degree in one unknown.

e.g., $x^2=9$. It follows that $x=\pm\sqrt{9}=\pm 3$.

That is, we get two solutions and each of them will check, since $(-3)^2=9$ and $(+3)^2=9$. Again $(x-2)^2=16$. Then $x-2=\pm 4$. Hence $x=2+4$ or $2-4$ and $x=6$ or -2. Both of these roots check.

Exercise 115

328. Solve and check:

1. $x^2=4$
2. $x^2=25$
3. $4x^2=36$
4. $4x^2=49$
5. $(3x)^2=16$
6. $(6x^2)=144$
7. $(x+1)^2=49$
8. $x^2+2x+1=49$
9. $(2x-3)^2=16$
10. $x^2+2x=48$
11. $x^2+x+1=49-x$
12. $2(3x-7)^2=32$
13. $x^2+4x+4=4$
14. $x^2+4x=0$
15. $x^2=3$
16. $(x-1)^2=5$

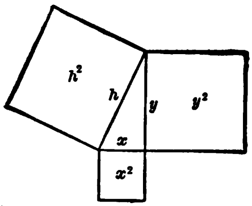
APPLICATION TO THE RIGHT TRIANGLE

329. The Theorem of Pythagoras. One of the oldest known facts of geometry, as well as one of the most useful, is the so-called Pythagorean theorem, viz.:

The sum of the squares of the legs of the right triangle equals the square of the hypotenuse.

In the adjoining figure $x^2+y^2=h^2$. Suppose $x=3$ ft.. $y=4$ ft., then $h^2=9+16=25$.

Therefore $h=\pm 5$.



The minus sign has no meaning here. Again, if $x=6$ ft. and $y=13$ ft., then $h^2=36+169=205$ and $h=\sqrt{205}=14.31+$ft.

Exercise 116

Find the third side of the following right triangles. If the result is not a perfect square extract the root to two decimal places.

1. $x=2$, $y=2$
2. $x=6$, $y=4$
3. $x=5$, $y=12$
4. $x=6$, $y=7$
5. $h=5$, $x=4$
6. $h=30$, $x=7$
7. $x=10$, $h=15$
8. $h=1$, $x=\frac{1}{2}$
9. $h=2$, $x=1$
10. $h=1$, $y=\frac{\sqrt{3}}{2}$

APPLICATION TO PROPORTION

(The student should study again §§ 258 to 264 on proportion.)

330. Definitions. A number x is said to be a *mean proportional* between a and b if

$$\frac{a}{x}=\frac{x}{b}$$

Since the product of the means equals the product of the extremes,

$$x^2=ab;\ \text{hence } x=\pm\sqrt{ab}.$$

x is also called the *geometric mean* between a and b.

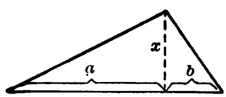
Example. Find a mean proportional between 2 and 8.

$$\frac{2}{x}=\frac{x}{8}$$

Multiply means and extremes $x^2=16;\ x=\pm 4.$

Check: $\frac{2}{4}=\frac{4}{8},\ \frac{2}{-4}=\frac{-4}{8}$

331. It is a well known theorem of geometry that the perpendicular dropped from the right angle of a right triangle on the hypotenuse is a mean proportional between the segments of the hypotenuse.



For example, find the length of the perpendicular from the vertex on the hypotenuse if the segments of the hypotenuse are 3 and 12.

Solution: $\frac{3}{x}=\frac{x}{12};\ x^2=36;\ x=\pm 6.$

The negative solution has no meaning here and hence is *discarded.*

Exercise 117

Find mean proportionals between:

1. 4; 9
2. 4; 16
3. 7; 15
4. 5; 125
5. 4; 25
6. 12; 1728
7. a; $4a$
8. $\dfrac{x^2}{2}$; 2
9. $\dfrac{3x^2a^3}{7}$; $21x^4a$

10. Find the length of the perpendicular from the vertex of the right angle on the hypotenuse if the segments of the hypotenuse are 18 and 2.

11. In problem 10 what is the perpendicular if the segments of the hypotenuse are 10 and 15?

12. Find the perpendicular if the whole hypotenuse is 21 and one segment is 9.

CHAPTER XVI

QUADRATIC EQUATIONS

INTRODUCTION

332. In Chapter XIII equations of one unknown were studied, while in Chapter XIV we considered simultaneous equations of two unknowns. Attention was then called to the fact that the equations solved were simple equations, that is, equations of the first degree in the unknowns.

The way has now been prepared for the solution of equations of the second degree in the unknowns. Such equations will now be considered first in equations of the second degree in one unknown and then in simultaneous equations of the second degree in two unknowns.

333. Definitions. An equation of the second degree is called a *quadratic equation*, or, more simply, a *quadratic.*

334. In an equation in one unknown, if both the first and second powers of the unknown occur, the equation is called a *complete quadratic* or an *affected quadratic.* Thus, $2x^2+4x-1=0$ is a complete quadratic.

335. An equation containing only the second degree of the unknown is called an *incomplete quadratic*, or, more commonly, a *pure quadratic.* Thus, $3x^2-27=0$ is a pure quadratic.

336. The pure quadratic was discussed in the last chapter. The affected quadratic is harder to solve and its solution is our chief concern in this chapter. Thus,

Solve $x^2-3x+2=0$

Factoring $(x-2)(x-1)=0$

The left member will vanish if $x=2$ or if $x=1$. Therefore, in order to get all possible roots, we place each factor equal to zero and solve it for x. Both these values check.

For example, solve $3x^2+4x+1=0$

Factoring $(3x+1)(x+1)=0$

Therefore $3x+1=0$ and $x=-\frac{1}{3}$

$x+1=0$ and $x=-1$

Check: $3(-\frac{1}{3})^2+4(-\frac{1}{3})+1=\frac{1}{3}-\frac{4}{3}+1=0$

$3(-1)^2+4(-1)+1=3-4+1=0$

Exercise 118

Solve the following by factoring and check:

1. $x^2+5x+6=0$
2. $x^2-6x+5=0$
3. $x^2+x-2=0$
4. $x^2+7x-30=0$
5. $2x^2+3x+1=0$
6. $4x^2+4x+1=0$
7. $5x^2-2x-3=0$
8. $9x^2+18x+1=0$
9. $6x^2+22x+20=0$
10. $16x^2-6x-5=-6+2x$

GRAPHICAL REPRESENTATION OF QUADRATIC EQUATIONS

337. Graphical Solution of Quadratic Equations. In §§ 288 to 292 was shown a method of plotting the graph of a linear equation. The method of plotting the graph of a quadratic equation is exactly the same.

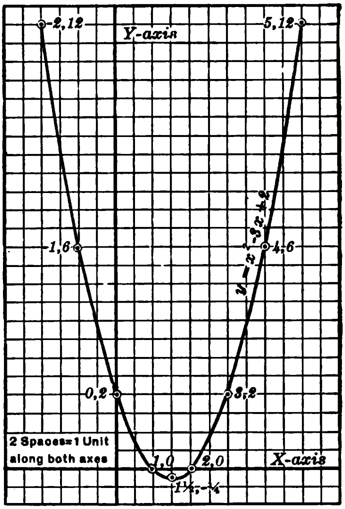
Suppose we have the equation

$$y=x^2-3x+2$$

Make a table of corresponding values of x and y as in § 284.

x	y
-2	12
-1	6
0	2
1	0
$1\frac{1}{2}$	$-\frac{1}{4}$

x	y
2	0
3	2
4	6
5	12



Now locate these points as was done in § 291 and connect the points: the result will be as shown in the figure. This curve is called the *graph* of the equation

$$y=x^2-3x+2$$

and the process is known as *plotting the graph* of that equation.

The student will notice that the graph is not a straight line, and this will be true of every quadratic equation, the first degree equation being the only equation whose graph is a straight line.

Now notice that at the points where the graph crosses the x-axis, *i.e.*, where $y=0$, the values of x are 1 and 2, and notice, too, that the roots of the equation as shown in § 336 are 1 and 2.

This, then, is a graphical method for solving a quadratic equation of one unknown, or of one variable, and may *be stated* as follows:

Plot the graph of the equation formed by writing y in place of the zero of the given equation. The distances from the origin to the points where the graph crosses the x-axis, with their proper signs, will be the roots of the equation. How many such points will there be in general?

Let the student plot and solve graphically all the equations of Exercise 118.

EXERCISES ON COMPLETING THE SQUARE

338. Before taking up these exercises the student should review § 213 on factoring perfect squares.

339. Since $(a+b)^2=a^2+2ab+b^2$ it follows that, in order to be a perfect square, a trinomial must have two terms that are squares of monomials and the remaining term must be twice the product of these monomials. Usually, however, the terms a^2+2ab are given and we wish to find what the last term must be. This can be done by taking half the coefficient of a and squaring it, as:

$$a^2+(2b)a+\left(\frac{2b}{2}\right)^2=a^2+2ab+b^2$$

Example 1. What must be added to x^2+4x to make a perfect square? Half the coefficient of x is 2 and $2^2=4$. Hence if 4 be added, the trinomial result is a perfect square $x^2+4x+4=(x+2)^2$.

Example 2. x^2-8x can be changed to a perfect square by adding $\left(\frac{8}{2}\right)^2=4^2=16$. Hence $x^2-8x+16=(x-4)^2$.

Example 3. $x^2-7.1x$ can be changed into a perfect square by adding $\left(\frac{7.1}{2}\right)^2=(3.55)^2=12.6025$.

Hence $x^2-7.1x+12.6025=(x-3.55)^2$.

It must be carefully noted that this method is valid only if the coefficient of x^2 is 1.

Exercise 119

Add the term necessary to make a complete square. Check by squaring.

1. x^2+2x

Add $\left(\frac{2}{2}\right)^2=1.\quad x^2+2x+1=(x+1)^2.$

Check:
$$\begin{array}{r} x+1 \\ \underline{x+1} \\ x^2+x \quad \\ \underline{\quad +x+1} \\ x^2+2x+1 \end{array}$$

2. x^2-2x
3. x^2+x
4. x^2+4x
5. a^2+3a
6. $a^2+.7a$
7. $(x+1)^2+7(x+1)$
8. p^2+14p
9. $x^2+1\frac{1}{2}x$
10. x^2-9x
11. x^2-120x
12. $b^2+2.2b$
13. $\left(\frac{b}{2}\right)^2+2.2\left(\frac{b}{2}\right)$
14. $x^2-3.20x$
15. $4x^2+4x$
16. $a^2+3.1416a$
17. x^2+ax
18. $x^2+\frac{bx}{a}$

SOLUTION OF QUADRATICS BY COMPLETING THE SQUARE

340. By transposing, collecting, and dividing by the coefficient of x^2 a quadratic equation can always be put into the form

$$x^2+\frac{bx}{a}+\frac{c}{a}=0$$

or
$$x^2+\frac{bx}{a}=-\frac{c}{a}$$

Example. $3x^2-7x-5=x^2-3x+25$

Transposing, $2x^2-4x=30$

Dividing, $x^2-\frac{4x}{2}=\frac{30}{2}$ or 15

This is in the form indicated above where $\frac{b}{a}$ is $\frac{-4}{2}$ and $\frac{-c}{a}$ is $\frac{30}{2}$.

A quadratic can be solved if the unknowns are grouped on the left side of the equality sign and made into a perfect square.

341. Model Solution.

Solve: $3x^2-7x-5=x^2-3x+25$

Transposing and collecting as above,

$$x^2-2x=15$$

Adding to complete the square $(\frac{2}{2})^2=1$ to both members,

$$x^2-2x+1=16$$
$$(x-1)^2=(4)^2$$

Hence $x-1=\pm 4$

$$x=1+4 \text{ or } 1-4$$
$$x=5 \text{ or } -3$$

Check: $3(5)^2-7(5)-5=(5)^2-3(5)+25$

$$75 \quad -35 \quad -5=25 \quad -15 \quad +25$$
$$35=35$$
$$3(-3)^2-7(-3)-5=(-3)^2-3(-3)+25$$
$$27 \quad +21 \quad -5=9 \quad +9 \quad +25$$
$$43=43$$

Exercise 120

Solve and check both roots in the original equation:

1. $x^2-6x=7$
2. $x^2-8x=+9$
3. $x^2+4x=12$
4. $x^2+3x=-x+5$
5. $2x^2+6x=4x+12$
6. $x^2-.2x=1.2$
7. $x^2+6.4x=.4x-9$
8. $3x^2-2x=20x+2x^2+23$
9. $x^2-\frac{3}{2}x=\frac{27}{2}$
10. $25-x^2=5x$

Solution of 10:

Transposing, $-x^2-5x=-25$

Changing all signs, $x^2+5x=25$

Completing the square, $x^2+5x+(\frac{5}{2})^2=25+(\frac{5}{2})^2$

$$(x+\tfrac{5}{2})^2=\tfrac{125}{4},\ x+2.5=\pm\sqrt{\tfrac{125}{4}}=\frac{\pm5\sqrt{5}}{2}=\frac{+5}{2}(2.236)$$

$$x=-2.5\pm5(1.118)=-2.5\pm5.590$$

$$x=3.09 \text{ or } -8.09$$

Check: $25-(3.09)^2=5(3.09)$; $25-9.548=15.45$; $15.452=15.45$
$25-(-8.09)^2=5(-8.09)$; $25-65.448=-40.45$; $-40.448=-40.45$.

When a radical is only approximated in the process of finding the square root one cannot expect a check to be absolute. The above is a good check for such an approximation.

11. $x^2-2x-4=0$

12. $2-6v=3v^2$

13. $8x+4=9x^2$

14. $a^2+10a+13=0$

15. $\frac{3x}{2}+4-\frac{1}{3x}=2\frac{1}{2}$

16. $\frac{5r}{2}-\frac{3}{r}-8=0$

17. $y-\frac{7}{y-3}=0$

18. $\frac{17}{s-5}-\frac{s}{2}=0$

19. $\frac{3}{s-1}-\frac{3+s}{3}=\frac{3}{5}$

20. $2x+3x^2=9$

21. $5p^2+15p=-2-p$

22. $\frac{3+x}{4+x}-\frac{x-5}{x-6}=\frac{1}{12}$

23. Make a formula for solving a quadratic equation.

24. $x^2+y^2-3x+7y+4=0$. If $y=-1$ solve for x. If $y=-2$ solve for x.

25. Solve problem 24 if $y=-4$; $y=-10$

26. $x^4+3x^2+2=0$

Solution: This may be regarded as a quadratic equation in which x^2 is the unknown.

Then $x^4-3x^2+\frac{9}{4}=-2+\frac{9}{4}=\frac{1}{4}$

$(x^2-\frac{3}{2})^2=\frac{1}{4}$; $x^2-\frac{3}{2}=\pm\frac{1}{2}$; $x^2=+\frac{3}{2}\pm\frac{1}{2}=+1$ or $+2$.

Therefore $x=\pm 1$ or $\pm\sqrt{2}$.

Check: $(1)^2-3(1)+2=0$; $4-6+2=0$.

27. $x^4-17x^2+30=0$

28. $4x^4-4x^2-2=x^2-3$

29. $9x^4-1=37x^2-5$

30. $4p^4+1=21p^2-4$

31. $(2x-3)(2x+3)=-(3x-1)^2$

32. $5x+9x^2+6(x-x^2)=2$

QUADRATICS WITH LITERAL COEFFICIENTS

342. Use of Literal Equations. In order to get the greatest advantage from the study of algebra it is necessary to be able to solve equations with letters in the coefficients. Often a great deal of labor is saved by being able to change an algebraic expression with literal coefficients into a form suitable for numerical calculation. The authors know of an instance in which an expert bookkeeper worked for a month on a problem in compound interest and finally gave it up in despair. It was solved in a few hours by a man who knew how to put his formula in the proper form before attempting to do any computing.

Example. The space traversed by a body falling from rest near the surface of the earth is given by the formula $s=\frac{1}{2}gt^2$ where s is measured in feet and t (time) in seconds.

Find the time (t) in terms of g and s.

Solution: Multiplying by 2 $\quad 2s=gt^2$.

Dividing by g, $\frac{2s}{g}=t^2$; hence $t=\pm\sqrt{\frac{2s}{g}}$

If $g=32$, find the time it takes a body to fall 64 feet. Answer, ± 2 seconds.

Has the − sign any meaning in this case?

Example. Solve $x^2+2x=ax+2a$.

Transposing, $x^2+(2-a)x=2a$.

Completing the square, $x^2+(2-a)x+\left(\frac{2-a}{2}\right)^2=2a+\left(\frac{2-a}{2}\right)^2$

$\left(x+\frac{2-a}{2}\right)^2=2a+\frac{4-4a+a^2}{4}$; $\left(x+\frac{2-a}{2}\right)^2=\frac{8a+4-4a+a^2}{4}$
$=\frac{4+4a+a^2}{4}$. Extracting root, $x+\frac{2-a}{2}=\frac{\pm(2+a)}{2}$

Transposing, $x=-\frac{2-a}{2}\pm\frac{2+a}{2}$; $x=\frac{-2+a+2+a}{2}$ or $\frac{-2+a-2-a}{2}$;
$x=a$ or -2.

Check: $a^2+2a=a(a)+2a$, or $a^2+2a=a^2+2a$

$$4+2(-2)=a(-2)+2a$$
$$4-4=-2a+2a;\quad 0=0$$

Exercise 121

Solve and check both roots:

1. $x^2+1=a^2+2x$
2. $x^2+4ax=5a^2$
3. $x^2+3x\sqrt{a}=4a$
4. $1-\frac{2}{x}+\frac{1}{x^2}=\frac{b^2}{4a^2}$
5. $x^2-2ax=9b^2-a^2$
6. $x^2+\frac{x(a+b)}{3}=ax+\frac{a+bx}{3}$
7. $\left(\frac{ay}{b}-\sqrt{a^2+b^2}\right)^2+y^2=b^2$
8. $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$
9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Solve for x; for y.
10. $\frac{a}{x}=\frac{x}{b}$. Solve for x.

PROBLEMS INVOLVING QUADRATICS

343. In solving a problem by quadratics always find what is the meaning of both roots (if possible). One root may have no meaning. If so, show a good reason for discarding it.

Exercise 122

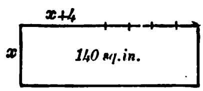
1. One number is 7 times greater than another and the sum of their squares is 337. Find the numbers.

2. A certain number added to its square equals 20. Find the number.

3. A certain number added to its square is 24.75. What is the number?

4. The length of a rectangle exceeds its breadth by 4 inches; its area is 140 square inches. Find its dimensions.

Whenever possible the student should draw a figure illustrating the problem. Thus:



5. One side of a rectangle is 2 feet longer than the other. The length plus twice the breadth equals the square of the length. Find the dimensions.

6. A number consists of two digits. The units' digit is the square of a number 2 greater than the tens' digit. The number itself equals 6 times the tens' digit plus 13. Find the number.

7. The area of a rectangular lot is 2400 square feet. Its perimeter is 280 feet. Find its dimensions.

SOLUTION OF QUADRATICS BY FORMULA

344. Use of Formula. A general formula which will solve every equation of a certain type is valuable as a labor saving device. It should not be used, however, till the student

thoroughly understands its meaning and can give its derivation.

345. The General Equation. The most general equation of the second degree in one unknown is

$$ax^2+bx+c=0$$

where x is the unknown and a, b, c are constants which may have any values—positive, negative, fractional, or even irrational. Every quadratic can be put into this form by clearing of fractions and collecting the coefficients of x and x^2. The solution of this general form will therefore give a formula that will hold in all cases.

$$\text{Dividing by } a,\ x^2+\frac{bx}{a}+\frac{c}{a}=0$$

$$\text{Transposing, } x^2+\frac{bx}{a}=\frac{-c}{a}$$

$$\text{Completing square, } x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\text{Simplifying, } \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a},\ \text{and } \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\text{Hence } x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\text{and } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{The two roots are } x_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\ \text{and } x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Exercise 123

Solve the following by substitution in the formula and check:

1. $6x^2+2x-1=x^2+7$

   Collecting, $5x^2+2x-8=0$

   Here $a=5$, $b=2$, $c=-8$

Hence $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{-2\pm\sqrt{4-4\cdot5(-8)}}{2\cdot5}$

$$x=\frac{-2\pm\sqrt{164}}{2\cdot5}=\frac{-2\pm2\sqrt{41}}{2\cdot5}=\frac{-1\pm\sqrt{41}}{5}$$

$$x=\frac{-1\pm6.40+}{5}=\frac{-7.40}{5}\text{ or }\frac{+5.40}{5}$$

$$x=-1.48+\text{ or }+1.08+$$

Check: $6(-1.48)^2+2(-1.48)-1=(-1.48)^2+7$

$13.14 \quad -2.96 \quad -1=2.19 \quad +7$

$9.18=9.19$

This is as close a check as can be expected when the radical is approximated to only two decimal places. Check the other root.

2. $x^2-5x-14=0$

3. $25x^2-30x=45$

4. $a^2+\dfrac{7}{12}a-1=0$

5. $\dfrac{7}{3y-2}+\dfrac{5}{3-2y}=-6$

6. $\dfrac{p}{p-1}+\dfrac{p+1}{p}=\frac{3}{4}$

7. $15x^2+4x=4-x^2+3x$

8. $3x-7=\dfrac{5x^2-4x+3}{2x+7}$

9. $1-6b^2=4b$

10. $x^2+3hx-4h^2=0$

11. $ax^2=-a^2+6x$

12. $x^2+2x=hx+2h$

13. $y^2+ay-by-ab=0$

14. $mx^2+x=\dfrac{3mx-3}{n}$

15. $x^2+x\sqrt{2}=3$

16. $\dfrac{\sqrt{a}}{x^2}+\dfrac{b}{x}=\dfrac{\sqrt{b}}{\sqrt{a}}$

17. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. Solve for x; for y.

18. $x^2+y^2=r^2$. Where $y=bx+b$. solve for b.

ANOTHER METHOD OF CHECKING CERTAIN SOLUTIONS

346. The two roots of $ax^2+bx+c=0$, § 345, are

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The subscripts merely call attention to the fact that there are two different values of x which satisfy the equation.

Adding, $x_1+x_2=\dfrac{-2b}{2a}=\dfrac{-b}{a}$

That is, *the sum of the two roots of any quadratic equals the negative of the coefficient of x divided by the coefficient of x^2.*

Again, if the two roots be multiplied, since the right members differ only in the sign of the radical,

$$x_1x_2=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}=\frac{+b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Therefore, *the product of the roots of any quadratic equals the known term divided by the coefficient of x^2.*

Example. $x^2+7x+4=0$.

Call the two roots x_1 and x_2.

Then $x_1+x_2=-\frac{7}{1}=-7$.

and $x_1x_2=\frac{4}{1}=4$.

Solving by formula

$$x=\frac{-7\pm\sqrt{49-16}}{2}=\frac{-7\pm\sqrt{33}}{2}$$

$$x_1=\frac{-7+\sqrt{33}}{2} \quad \text{and} \quad x_2=\frac{-7-\sqrt{33}}{2}$$

The sum of the roots is -7. The product $x_1x_2=\dfrac{49-33}{4}=\dfrac{16}{4}=4$.

This method serves as a check on the roots and especially on roots involving radicals. In many cases it is much shorter than the extraction of the arithmetical root. It has the further advantage of being exact, while the arithmetical ex-

traction of a root is only approximate. It has the disadvantage, however, of not checking the preliminary work, which is sometimes needed to get the equation into the standard form.

Exercise 124

Solve by formula and check by § 346:

1. $2x^2+7x-14=(x-1)^2$
2. $3x^2-2x+6=6(x+3)^2$
3. $(x-1)^2+3(x+2)^2=7x+23$
4. $6x^2-13x+7=3\frac{1}{2}$
5. $(x-1)^2+7(x-1)+3=2(x-1)$
6. $31x-74x^2+13=0$
7. $9.4x-2.1x^2-4.4=0$
8. $19.2x+3.2(x-.2)^2=7.12$

Exercise 125—Problems Involving Quadratics

1. Using the formula $s=16t^2$ where s is measured in feet and t in seconds, find how long it will take a ball to fall 1900 feet.

2. The Washington Monument (Washington, D. C.) is 555 feet high. How long will it take a ball to fall from the top to the ground?

3. In problem 2 if the formula for velocity is $v=32t$, with what velocity will the ball strike the ground?

4. An aviator drops a number of bombs from a height of 2000 feet. With what velocity will they strike the ground?

5. If three times the square of a number is increased by the number itself the sum is 10. Find the number.

6. The sum of the squares of two consecutive numbers is 421. Find the numbers.

7. Can you find the numbers in problem 6 if the sum of the squares is 213?

8. The square of a number plus 1 divided by that number equals $\frac{37}{6}$. Find the number.

9. If the product of two consecutive numbers, increased by 4, be divided by their sum the result is 4. Find the numbers.

10. The numerator of a certain fraction exceeds the denominator by 4. If the fraction is inverted its value is decreased by $\frac{40}{21}$. Find the fraction.

11. The numerator of a fraction is 6 less than the denominator. If 5 be added to the numerator and denominator the fraction equals $\frac{2}{3}$. Find the fraction.

The formula $s=-16t^2+vt$ gives the space traversed by a body thrown up with speed v. Like the formula $s=16t^2$ the s and t are measured in feet and seconds, but the positive direction is taken as up now instead of down as before. This accounts for the minus sign before $16t^2$.

12. If v is 100 feet per second how high will a ball be in 4 seconds?

Solution: $s=-16(4)^2+100(4)=-256+400=+144$ feet high.

13. If v is 100 feet per second, how high will the ball be in 6 seconds?

14. If v is 100 feet per second, how high will the ball be in 7 seconds?

The negative sign in problem 14 means that the ball has gone up to its highest point, then fallen back past the starting place and 84 feet below it. If the ball were thrown from the ground this is of course impossible. But somewhere between 6 and 7 seconds the ball reached the ground again. The student can find exactly how long it will take the ball to come back to the starting place by making $s=0$ and solving for t, thus:

$$0=-16^2+100t$$

Transposing, $16t^2-100t=0$, and factoring, $t(16t-100)=0$.

$$\text{Solving } t=0$$
$$t=\frac{100}{16}=6\tfrac{1}{4}, \text{ seconds}$$

15. What is the meaning of the solution $t=0$ in problem 14?

16. It is evident that it took the ball as long to reach the highest point as it was in falling from that point back to the ground. How long was it in going up? How long coming down?

17. From problem 16 find how high the ball will go.

18. Throw a ball upward, and count the seconds till it reaches the ground again; give data enough to find how high it went. Who can throw it the highest?

19. A boy standing on the county bridge at Knoxville dropped stones into the Tennessee River with speed 0. They reached the water in $2\frac{1}{2}$ seconds. How high is the bridge?

20. A boy throws a walnut up with velocity of 60 feet per second. When it is 50 feet high how long has it been in motion? What do the double answers mean?

21. A rifle shoots a bullet straight up with muzzle velocity of 2000 feet per second. When it is 1500 feet high how long has it been in motion?

22. In problem 21, how long till the bullet reaches the ground again?

23. In problem 21, how high did the bullet go?

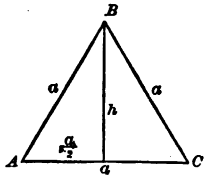
24. Solve formula $s=-16t^2+vt$ for t in terms of s and v.

25. A body is thrown up with velocity 20 feet per second. How long has it been in motion when it is 14 feet high? Are there two answers? If so, what do they mean?

Exercise 126—Geometrical Applications

An equilateral triangle (§ 144) has all three of its sides equal in length. Let ABC be an equilateral triangle with

each side a feet in length. It is a well-known fact of geometry that in an equilateral triangle the perpendicular from any



vertex on the opposite side cuts that side in halves. In the right triangles formed, the hypotenuse, a, and one leg, $\frac{a}{2}$, are known.

1. Find h in terms of a by the relation $(\text{hypotenuse})^2 =$ sum of the squares of the two legs. Or briefly $h^2=b^2+p^2$.

2. If a side of an equilateral triangle is 10 feet, find the altitude h.

3. The formula for the area of any triangle, whether equilateral or not, is

$$A=\frac{\text{base}}{2}\times\text{altitude (§158), or more briefly, } A=\frac{bh}{2}.$$

Find the formula for the area of an equilateral triangle.

Ans. $A=\dfrac{a^2\sqrt{3}}{4}$.

4. If a side of an equilateral triangle is 12 feet find the area by the formula of exercise 3.

5. From $h^2=b^2+p^2$ find the unknown legs of the following right triangles, correct to 2 decimal places:

$p=3,\ b=4$	$b=5,\ h=7$
$p=6,\ b=8$	$b=5,\ h=17$
$p=7,\ b=12$	$h=9,\ p=8$
$p=7,\ b=10$	$h=15,\ b=14$

6. From the formula of exercise 3 find correct to two decimal places the sides of the equilateral triangles whose areas are:

$A=24$ square feet	$A=17.5$ square feet
$A=\frac{4}{3}$ square inches	$A=9.41$ square feet
$A=165$ square miles	$A=127.44$ square feet

SIMULTANEOUS QUADRATICS

347. Definition. If two or more quadratics with two or more unknowns appearing in them are so related that the same set of values of the unknowns will satisfy all of the equations they are called a system of *simultaneous quadratics*.

The pupil will notice that this definition is almost word for word the same as that for simultaneous simple equations, § 283.

348. Method of Solving. When one equation is quadratic and the other is linear, the roots of these two equations can always be found by solving the linear equation for one of the unknowns in terms of the other, and substituting this value in the quadratic.

Example. Find the roots of the two simultaneous equations

(1). $x^2+3y=27$

(2). $3x+y=15$

Solution: Find value of y from equation 2.

(3). $y=15-3x$

Substitute this value for y in equation 1.

(4). $x^2+3(15-3x)=27$

Remove parentheses.

(5). $x^2-9x+45=27$

Or $x^2-9x+18=0$

Solving by formula § 345.

$$x=\frac{9\pm\sqrt{81-72}}{2}=6 \text{ or } 3$$

From equation (3) $y=-3$ or $+6$

Checking, $x=6,\ y=-3$

(1). $36-9=27$

(2). $18-3=15$

Checking, $x=3,\ y=6$

(1). $9+18=27$

(2). $9+6=15$

Exercise 127

Solve and check the following pairs of equations:

1. $x^2+y^2=1$
   $3x-2y+2=0$
2. $x^2+2y^2-2x=26$
   $x+y=7$
3. $x^2+3xy=10$
   $2x+y=5$
4. $3x^2-y^2=3$
   $\frac{x}{2}+y=2$
5. $xy-2x=6$
   $y+2x=9$
6. $3x^2+5y^2=23$
   $\frac{x}{2}-\frac{y}{4}=0$
7. $x^2+y^2-2x+y=12$
   $\frac{x}{4}+\frac{y}{6}=\frac{5}{6}$
8. $xy-3y^2+x=12$
   $x-2y-4=0$
9. $3xy=36$
   $2x-5y=2$
10. $3x^2+5y=17$
    $5x^2-y=19$
11. Solve for a
    $a^2+ab-b^2+2a-b=15$
    $2a+b=4$
12. Solve for a
    $a^2+ab-5b=235$
    $2a+3b=33$

349. Graphical Solutions. The equations in the mod solution of § 348 can be solved graphically as follows:

From Equation 1, finding the value of y in terms of x,

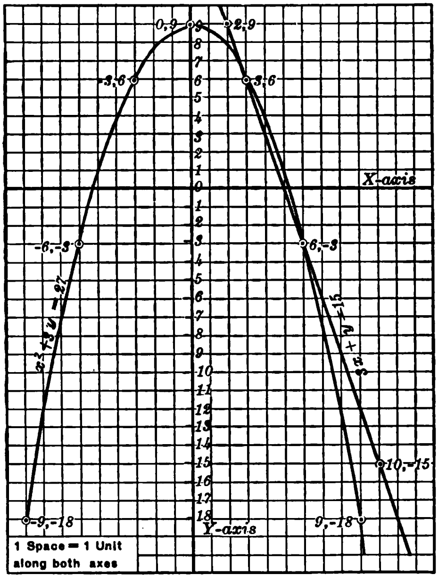
$$y=\frac{27-x^2}{(3)}$$

Make a table of corresponding values of x and y as in § 284 and in § 337.

x	y
± 9	-18
± 6	-3
± 3	6
0	9

x	y
3	6
6	-3
9	-21

Plot the graph of the equation by connecting the points



determined by these pairs of numbers. See the figure.

Then in the same way plot the graph of Equation 2 with respect to the same pair of axes.

You will notice that the two graphs cross at points whose distances from the axes are 3, and 6 from the y-axis and 6 and -3 from the x-axis.

We have then a method for the solution of a quadratic and a linear equation identical with that given in § 296 for the solution of two linear equations.

Most of the exercises given in the preceding list are rather difficult for you to solve by this method without further discussion of graphs.

Problems 1, 4, 9, and 10 of Exercise 127 you should be able to solve after a little study. Do them by both the substitution method given in § 348 and by the graphical method just shown, and compare results.

If your plotting is carefully done the results should agree exactly.

350. Equations in One Variable. An application of the method shown in § 349 gives an extremely convenient graphical solution for quadratic equations of one variable.

For example: consider the equation solved in § 336

$$x^2-3x+2=0$$

To solve this graphically

Let $y=x^2$

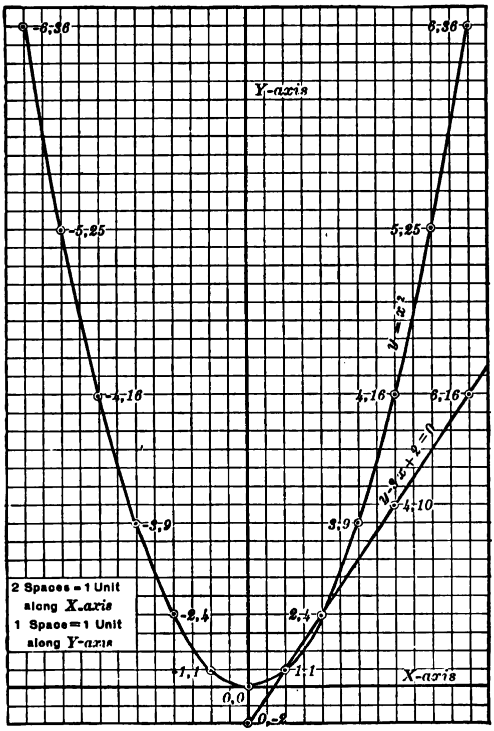
Plot the graph of this equation by making a table of corresponding values of x and y as in other exercises, and connecting the points thus determined.

x	y
± 0	0
± 1	1
± 2	4
± 3	9
± 4	16
± 5	25
± 6	36

Note. A much better graph can be made by taking, say, two squares for one unit along the x-axis and one square as a unit along the y-axis than by making the scale the same along both axes. The graphs on *the opposite* page are made to this modified scale.

After plotting the graph for equation $y=x^2$, substitute in the given equation, $x^2-3x+2=0$, y for the x^2, which gives $y-3x+2=0$.

This is a linear equation, having a straight line for its graph (§ 294).



Plot the graph of this equation (see the above figure) with respect to the same pair of axes as for equation

$$y=x^2$$

being careful to use the same scale of units.

Then the distances of the points of intersection of the graph of

$$y=x^2$$

with that of

$$y-3x+2=0$$

from the y-axis with their proper signs will be the roots of the equation

$$x^2-3x+2=0$$

Exercises. Solve the equations of Exercises 118 and 120 by this method, checking your results by some other method not graphical.

SPECIAL METHODS

351. When both equations are quadratics the methods of solution are quite arbitrary and cannot be stated in a general law. Below are given, however, methods for the solution of two types of simultaneous quadratics:

Example 1. Find roots of equations:

$$(1)\quad 4x^2+9y^2=585$$
$$(2)\quad 8xy=336$$

Dividing (2) through by 8 gives (3) $xy=42$.

An inspection of (1) shows that if there were added to the left member $12xy$, that member of the equation would be a square, *i.e.*, would have the form

$$(3)\quad 4x^2+12xy+9y^2$$

Multiplying (3) by 12 and adding result to (1) gives

$$(4)\quad 4x^2+12xy+9y^2=1089$$

Extracting square root of both members of (4) gives

$$(5)\quad 2x+3y=\pm 33$$

This being a simple equation (5) and (2) or (5) and (1) may be solved according to the method first given for simultaneous quadratics.

Example 2. Find roots for equations:

$$(1)\quad 2x^2-3y^2=23$$
$$(2)\quad x^2+2y^2=43$$

These two equations can be combined as in simultaneous simple equations.

If equation (2) is multiplied by 2 and the result taken from equation (1) we get $\quad (3)\ -7y^2=-63$

Therefore, $\quad y^2=9$

$$y=\pm 3$$

Substituting for y in (2)

$$x^2+18=43$$
$$x^2=25$$
$$x=\pm 5$$

Check: $x=5$, $y=3$ in (1) $\quad 50-27=23$

(2) $\quad 25+18=43$

Exercise 128

Solve and check the following pairs of equations:

1. $3x^2-7y^2=47$
   $x^2+y^2=29$
2. $4x^2+y^2=13$
   $2xy=6$
3. $x^2-3xy+4y^2=11$
   $xy=10$
4. $3x^2+y^2=21$
   $x^2-y=1$
5. $\dfrac{x^2}{4}-\dfrac{2y^2}{3}=\dfrac{1}{4}.$
   $x^2-2y^2=7$
6. $5x^2-4y^2=61$
   $10x+2y^2=82$
7. $2xy=12$
   $x^2+9y^2=45$
8. $4x^2+3xy=34$
   $xy+y^2=15$
9. $x^2-8xy=9$
   $2xy+9y^2=27$
10. $3x^2+5y=17$
    $5x^2-y=19$

352. Summary of Quadratic Equations. There are many ways of solving quadratic equations, but the most important of them are the following:

1. Simplify and transpose all terms to the left member. Then factor and set each factor equal to zero. Solve these two equations.

2. If the factors of the left members are not easily found solve by completing the square.

3. For equations of one variable:

Plot the graph of equation $y=x^2$, then substitute y for x^2 in the given equation and plot the resulting equation to same scale and same axes as the graph of $y=x^2$.

The values of x for the points of intersection of these graphs will be the solutions of the equation.

4. For simultaneous equations of two variables:

Plot the graphs of the two equations: the distances of their points of intersection from the axes, with their proper signs will be the solutions of the two equations.

5. If the student is well versed in completing the square he may solve any quadratic by the general formula of § 345.

In the sections on simultaneous quadratics only the most important methods have been given. The special methods in common use are so varied that no summary of them would be profitable here.

Exercise 129 — Review Examples and Problems

The student is urged to read again the general directions for solving the problems in § 78 before attempting the following exercises and problems.

Solve and check:

1. $5x^2-7x\div 14=0.$
2. $a^2+18a=40$
3. $(2x+3)^2+18(2x+3)=40$
4. $\sqrt{3x^2+8}=\sqrt{5x+7}.$

Square both members

5. $\sqrt{\dfrac{2x+3}{2x-3}}=\sqrt{\dfrac{2x-7}{7x-2}}$
6. $x+\sqrt{x+1}=2\sqrt{x+1}$
7. $x^2+px+q=0$
8. $\dfrac{x-1}{x+1}+x=\dfrac{x(x-1)}{x^2-1}$

9. $2x^4-17x^2=9$

10. $3\sqrt{x}+x+2=0$

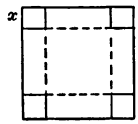
11. $16p^2-7p=6-2(p^2-6)$

12. The length of a rectangular piece of ground exceeds the width by 6 feet. A walk 6 feet wide surrounds the plot. The area of the walk is 12 times the area of the plot. Find the dimensions. Draw figure.

13. One side of a rectangle is 3 feet longer than the other. The diagonal is 5 feet. Find the area.

14. Six times a certain number added to twice the square of the number equals 36. Find the number.

15. A square piece of tin 18 inches on each side has a small square of side x cut out of each corner as shown. It is then folded along the dotted line so as to form an open box of volume $16x$. Find the side of the little square cut out.



16. The Masonic Temple in Chicago is 384 feet high. How long will it take a body to fall from the top to the ground?

17. With what velocity must a baseball player throw a ball to make it go 384 feet high? Can it be done?

18. Solve:

$$\frac{2}{x-6}=\frac{8}{3(x-1)}-\frac{4x-4}{3x(x-1)}$$

19. $\dfrac{3z-1}{7-z}=\dfrac{8+2z}{2z+1}$

20. $\dfrac{2}{1-a^2}+\dfrac{2}{1+a}=\dfrac{2}{1-a}-\dfrac{7}{4}$

21. Find the side of an equilateral triangle if the area is $\frac{80}{3}$ square feet.

22. Find the dimensions of a rectangle of area 1250 square feet if the perimeter is 160 feet.

23. A farmer is plowing a rectangular field whose dimensions are 440 yards by 220 yards. How wide a border must he plow to be half done? Three-quarters done?

24. In problem 23, if the plow cuts 12 inches how many rounds will the farmer have to go?

25. A tank can be filled by two pipes running together in 4 hours. The larger pipe alone fills the tank in 2 hours less time than the smaller. How long will it take each pipe to fill the tank?

26. In cutting the wheat in a square field of 40 acres how wide a strip must a farmer cut to be half done? One-third done? If the binder cuts a 6-foot swath how many rounds must he go to cut 20 acres?

27. A chest 24 inches long will just permit a 28-inch shotgun barrel to lie diagonally along the bottom. How long a chest of the same width will similarly accommodate a barrel 2 inches longer?

28. A certain athletic field is 380 feet long. It is desired to purchase a strip along one side so as to add a running track. If land costs $1000 per acre and $400 is available, how wide a strip can be bought?

29. A city block is 400 by 300 feet. At 3 cents a square foot it costs $228.30 to lay a walk around it. Find the width of the walk to the nearest inch.

30. If a rectangular lot containing 1200 square yards has its width increased by 2 yards and its length by 10 yards its area is increased by 400 square yards. What are the dimensions of the lot?

31. What values of x will satisfy equation

$$\frac{x+1}{c}-\frac{2}{cx}=\frac{x+2}{ax-bx}?$$

32. The product of two consecutive numbers exceeds their sum by 991. What are the numbers?

33. The difference between two numbers is 8 times the smaller, while the square of the smaller is $\frac{4}{9}$ the greater. Find the numbers.

34. Determine values of x satisfying the equation

$$\frac{2x-1}{x+3}+\frac{2(x+1)}{x+2}=\frac{25}{12}.$$

35. Find values for x and y satisfying the equations

$$x^2+y^2=52$$
$$xy+24=0$$

36. If the product of two numbers is increased by their sum the result is 79. If their product is diminished by their sum the result is 41. Find the numbers.

37. A farmer rents a certain number of acres for \$120.00. He subrents all but 6 acres for \$120.00 and receives \$1.00 per acre more than he paid per acre. How many acres does he rent?

38. The length of a field exceeds its breadth by 20 rods. If the dimensions be increased by 20 rods each way the area will be increased by 4800 square rods. Find dimensions of the field.

39. Find values of x satisfying equation

$$(x+m)^2+(x-m)^2=5mx$$

40. The difference between two numbers is one half the smaller; the sum of the squares is 326. Find the numbers.

41. Solve the following equation for the value of x:

$$\frac{x-5}{x+3}+\frac{x-8}{x-3}=\frac{80}{x^2-9}+\frac{1}{2}$$

42. The square of the sum of two consecutive numbers exceeds the sum of their squares by 40. What are the numbers?

43. A farmer bought a flock of sheep for $80.00. He kept 5 and sold the remainder for $90.00, thereby gaining $2.00 on each sheep sold. Determine the number of sheep in the flock.

44. A man sold a cow for $32.00 and gained in the trade one half as many percent as the cost in dollars. Find the cost of the cow.

45. A number is formed of two digits of which the units' digit is double the tens' digit. If to the sum of their squares 18 be added the order of the digits will be reversed. What is the number?

46. What values of x satisfy the equation

$$\frac{x+1}{x^2-4}+\frac{1-x}{x+2}=\frac{2}{5(x-2)}?$$

47. A rectangular field has a perimeter of 270 rods and an area of 4500 square rods. Find its dimensions.

48. A trader sells a number of horses for $2000.00. If he had sold 2 fewer horses for $15.00 more each he would have received $70.00 more for the lot. How many horses were sold?

49. If a train moved 6 miles per hour faster it would require 3 hours less to go 140 miles. Find its speed.

50. A number containing two digits is seven times their sum and the sum of their squares is 20. What is the number?

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